AN INDEPENDENT APPRAISAL OF THE POLARIS OPERATIONAL TESTING PROCEDURES

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AN INDEPENDENT APPRAISAL

OF THE POLARIS

OPERATIONAL TESTING PROCEDURES

by

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B. S., University of South Carolina, 1959

[redacted]

Submitted in partial fulfillment

for the degree of

MASTER OF SCIENCE IN OPERATIONS RESEARCH

from the

UNITED STATES NAVAL POSTGRADUATE SCHOOL

May 1966

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ABSTRACT

The first section of this paper is concerned with determining the required lower confidence limit that must be met by testing after a missile system becomes operational. Some of the costs of making decisions about the required system reliability lower confidence limit are discussed. Two possible cost effective models for determining the optimum test size are suggested.

The second section of this paper is concerned with the effects of changing the number of missiles tested in each year while maintaining the total number of systems tests constant over the missile system's projected operating life. In other words, in this section, the effects of testing heavily in the first years versus testing heavily at the end of the system's life or versus testing uniformly throughout the life of the system are compared. For this comparison, the sum of the variances of the estimators determined from the results of the tests conducted in each year is obtained. This value is compared for six representative distributions of testing throughout an estimated system life of ten years.

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1. INTRODUCTION

There are three types of tests for our current Polaris missile system:

1. The Demonstration and Shakedown Operation - DASO;
2. Operational Tests - OT;
3. Follow-on Operational Test - FOT.

DASO represents the first test of the weapon system in its totality. For this test, the missiles are carefully prepared and then fired under strigent conditions. The results of DASO firings are not included in the data for determining the weapon system reliability.

The OT program is a one-time test program with the specific objective of determining weapon system reliability. It is conducted under the most realistic conditions possible. The system is tested in such a way that the entire operational system, including communications, authentication, navigation, and accuracy, is tested; and the observed success rate is computed.

The FOT is conducted in the same manner as the OT with the specific objective of insuring that the weapon system reliability factor determined in the OT program is, in fact, still valid in the years following the Operational Tests.

This paper is concerned with determining the optimal OT and FOT test sample sizes. Much of the work presented here was started this summer at the Office of Programs Appraisal under the direction of Captain D. A. Paolucci. In addition, some of the ideas introduced come from unpublished notes of Captain Paolucci.

2. SUMMARY AND CONCLUSIONS

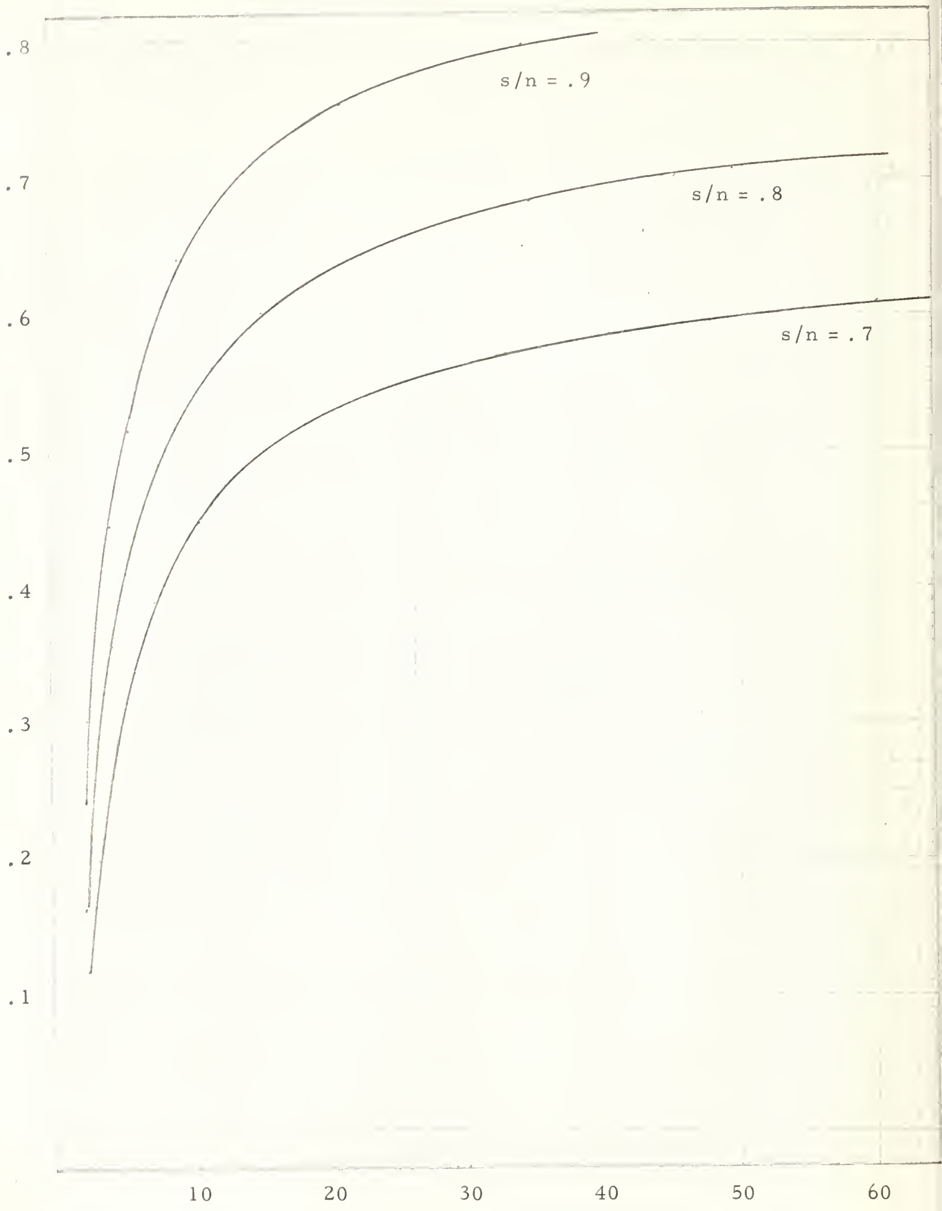
2.1 A Model for Maximizing the Number of Reliable Missiles in Inventory

In the final analysis, the number of missiles from a system counted on by the strategic planner will be determined by the number of missiles available and the 90 percent lower confidence limit[1] for system reliability. In other words, if the 90 percent lower confidence limit of reliability is .7 and there are N missiles available, the strategic planner will count on .7N when targeting the system.

Assuming that the observed success rate is relatively constant, the lower confidence limit of reliability will increase every time a missile is successfully tested. By observing Figure 2.1.1, it can be seen that the marginal return for each test reaches a near zero point when the lower confidence limit comes within .11 to .09 of the observed success rate. This observation might lead to the criterion that testing should continue until the lower confidence limit of reliability is within .11 to .09 of the observed success rate.

It can be shown, see section 3.1, that using this criterion for determining test sizes can, under certain circumstances, result in a

[1] To avoid messy notation and confusing the problem, this paper will always use the 90 percent confidence level. This choice of 90 percent is completely arbitrary and may not necessarily reflect Navy policy.



90 PERCENT LOWER CONFIDENCE LIMIT
VERSUS SIZE OF TEST (n)

FIGURE 2.1.1

smaller number of reliable missiles counted on by the targeter than would be counted on if fewer missiles were tested.

For this reason, the following alternate model for determining the required test size to estimate missile system reliability is suggested. Choose n to maximize Q in the following expression.

$Q = L(M - n)$, where:

L = the 90 percent lower confidence limit determined by n and the observed success rate;

M = the total number of missiles purchased;

n = the number of trials.

Using this criterion to determine the test size results in testing about 33 missiles if the observed success rate is in the area of .7 to .9.[1] The criterion mentioned earlier requires approximately 35, 46, and 49 missiles for observed success rates of .9, .8, .7, respectively.

A cost effective model for determining the test size to most effectively increase the reliable yield of the expected number of missiles deployed is presented in section 3.2. This model uses the marginal cost of increasing the expected yield of the Polaris missile system by procurement as the cut-off point for spending on missile testing.

The cost information necessary to obtain an explicit number for test size using this model is not known to the author. Therefore, no

[1]See section 3.1 for mathematical justification of this statement.

further discussion of this model will be presented in this section except to note that the figures necessary to obtain an explicit solution are obtainable.

2.2 Brief of Method Used in Investigating Models One, Two, and Three

The present doctrine used to determine FOT test sizes is to use the same size test that is necessary for the OT. Therefore, for the next ten years somewhere between 300 and 400 missiles would be programmed for expenditure during OT's and FOT's.

To investigate some of the effects on reliability estimation by changing the yearly FOT test size, six different sample plans that distribute the testing throughout the years in different ways were considered. These sample plans are given in Table 2.2.1.

To avoid confusion in this section, all testing done in year i will be designated as test i; and the individual shots will be designated as trials. Each different way of distributing the tests among the years will be called a sample plan.

To compare the different sample plans, the sum of the variances computed in each year will be obtained for each sample plan. In other words, the variance of the estimator of reliability will be obtained under a sample plan for year one. Then, using the same sample plan, the variance of the reliability estimator will be obtained for years two through ten. Once these values have been obtained, the ten of them will be added together and the sum will be called the sum of the variances.

The reason for the criterion of comparing the sum of the variances

Sample Plan No.	Year 1	Year 2	Year 3	Year 4	Year 5	Year 6	Year 7	Year 8	Year 9	Year 10
1	.1N	.1N	.1N	.1N	.1N	.1N	.1N	.1N	.1N	.1N
2	.05N	.05N	.05N	.05N	.05N	.15N	.15N	.15N	.15N	.15N
3	.15N	.15N	.15N	.15N	.15N	.05N	.05N	.05N	.05N	.05N
4	.3N	.1N	.1N	.1N	.1N	.05N	.05N	.05N	.05N	.05N
5	.05N	.05N	.05N	.05N	.3N	.1N	.1N	.1N	.1N	.1N
6	.3N	.2N	.1N	.1N	.1N	.1N	.03N	.03N	.02N	.02N

(N = the total number of missiles tested in the first ten years of operation.)

YEARLY TRIALS IN SAMPLE PLANS ONE THROUGH SIX

FIGURE 2.2.1

can be likened to the quest for a "minimum variance estimator" in many conventional estimation problems.

The sum of the variances depends on the different test sizes, the initial reliability (p_0), and the total number of missiles tested in the ten years. An attempt has been made to investigate the different sample plans over a plausible range of values for p_0 and N, the number of missiles scheduled for testing during the first ten years of operational life. To this end, the sum of the variances for each sample plan is computed using all possible combinations of .7, .8, .9 for p_0, and 200, 300, and 400 for N.

Perhaps a little more explanation is merited here. Assume that the sum of the variances of sample plan one is being computed. Now look at the situation where $p_0 = .7$ and $N = 200$. When $N = 200$, sample plan one requires that 20 trials of the system be conducted in the first year. From this, an expression for the variance of the estimator can be obtained. Likewise, an expression for the variance of the estimator in years two through ten can be obtained. Once the variances for all the years have been obtained, they are added together and the number is recorded as the sum of the variances under sample plan one, where $p_0 = .7$ and $N = 200$. Next, sample plan two will be used; and the sum of the variances under this sample plan, when $p_0 = .7$ and $N = 200$, will be obtained.

We will continue in this fashion until the sum of the variances when $p_0 = .7$ and $N = 200$ have been determined for sample plans one through six. Then, the next combination of p_0 and N, say $p_0 = .7$, $N = 300$, will be considered. In the same manner as before, the sum of the variances for all the sample plans will be obtained. The computations will continue until the sum of the variances for each sample plan has been obtained for all possible combinations of the three values of p_0 and N. Having obtained these variances, those sampling plans with the associated smaller variance values would be superior to those with larger variances from the point of view of having accurate estimates of missile reliability.

2.3 Determining FOT Sample Size Using Model One

Consider a collection of N missiles that are characterized by lifetimes $X_1, X_2, \ldots, X_N$ where X_i represents the time from initial inspection and release until the i^{th} item deteriorates to an unacceptable state. If it is assumed that the initial states of the missiles are the same, then the X_i's are non-negative independent, identically distributed random variables. Suppose that there are ten distinct times that observations are to be taken from the collection of missiles. Also, assume that K_i items are to be observed at time t_i. These observations can be summarized by the random variables $Y_1, Y_2, \ldots, Y_{K_i}$ where:

$$Y_i = \begin{cases} 1 & \text{if the missile system fails} \\ 0 & \text{if the missile system is successful} \end{cases}$$

On the basis of this information, it is desired to estimate $R(t_j)$, P (a missile's lifetime is greater than t_j), for each $t_j = j$, $j = 1, 2, \ldots, 10$. The estimation procedure which is proposed has the following form. For each j, the estimate of $R(t_j)$ is given by

$$R(t_j) = \frac{\prod_{i=1}^{j} (N - n_{i-1} - \sum_{K=(n_{i+1}1)}^{n_i} Y_K}{N(N-n_1)(N-n_2)\ldots(N-n_{j-1})},$$

where:

$$n_0 = 0, \quad n_i = \sum_{i=1}^{i} K_r, \quad n_i \quad 1, 2, \ldots, 10$$

$$K_r = \text{no. of trials in year } r$$

$$Y_i = \begin{cases} 1 & \text{if test } i \text{ fails} \\ 0 & \text{if trial } i \text{ is successful} \end{cases}$$

This estimator provides continuity to the estimates of reliability over the ten years, see section 3.3. However, this estimate is generally more optimistic in the early years and more pessimistic in the out years than the Maximum Likelihood Estimator. The MLE mentioned here is the MLE of reliability using only the information obtained from the current test. Note that Model One does take past data into account.

To investigate Model One, it was assumed that the probability of a successful trial at any given time t could be expressed by the exponential function:

$$P(t) = p_0 e^{-bt} \quad \text{where } b \text{ is an unknown parameter.}$$

The sum of the variances was then computed for all combinations of p_0 and N, as explained before. In addition, these values were computed for values of b ranging from .002 to .2. The results of these computations are displayed in the graphs on pages 20 through 37.

On these graphs, values of b are plotted along the abscissa, and values of the sum of the variances are plotted along the ordinate.

The grid superimposed on each graph is a one-inch square grid. The scale of the abscissa, x scale, the scale of the ordinate, Y-SCALE, the initial reliability, P(0), and the total number of trials in ten years, N, are lettered at the bottom of each graph. These values are written in powers of ten. The number directly next to the letter E is the power of ten by which the main number is to be multiplied. For example, the number 5.00E-01 is read as 5.0×10^{-1} or .5. The number 2.00E+02 is read as 2.0×10^{2} or 200. The numbers just to the right of each plot designates the sample plan that was used to obtain the plot. Suppose the sum of the variances for sample plan one is desired when $p_0 = .7$, $N = 200$, and $b = .1$. The graph for this case is found on page 20. The value desired can be read off the curve labeled 1. Similarly, the sum of the variances for sample plan six with the same parameters can be read from the curve labeled 6. To insure that the reader has found the proper values, these values have been marked with an x. The graphs are filed in the following order: all combinations of N with $p_0 = .7$, then all combinations of N with $p_0 = .8$, and then all values of N with $p_0 = .9$.

From reading these graphs, it can be seen that for all values of the parameters sample plan six and four have the smallest sum of the variances. The rest of the sample plans range from next smallest to largest value of the sum of the variances in the following order:

three, one, two, five. For ease in understanding the results, the table explaining the yearly test sizes for each sample plan is reproduced on the following page.

These results indicate that if 200 missiles were programmed for Follow on Testing in the next ten years there may be some merit to distributing the testing as shown below.

> Suggested allocation per year of FOT testing if a total of 200 trials are to be conducted:
>
> Year one -- 60 trials;
>
> Years two through six -- 20 trials;
>
> Years seven through ten -- 10 trials.

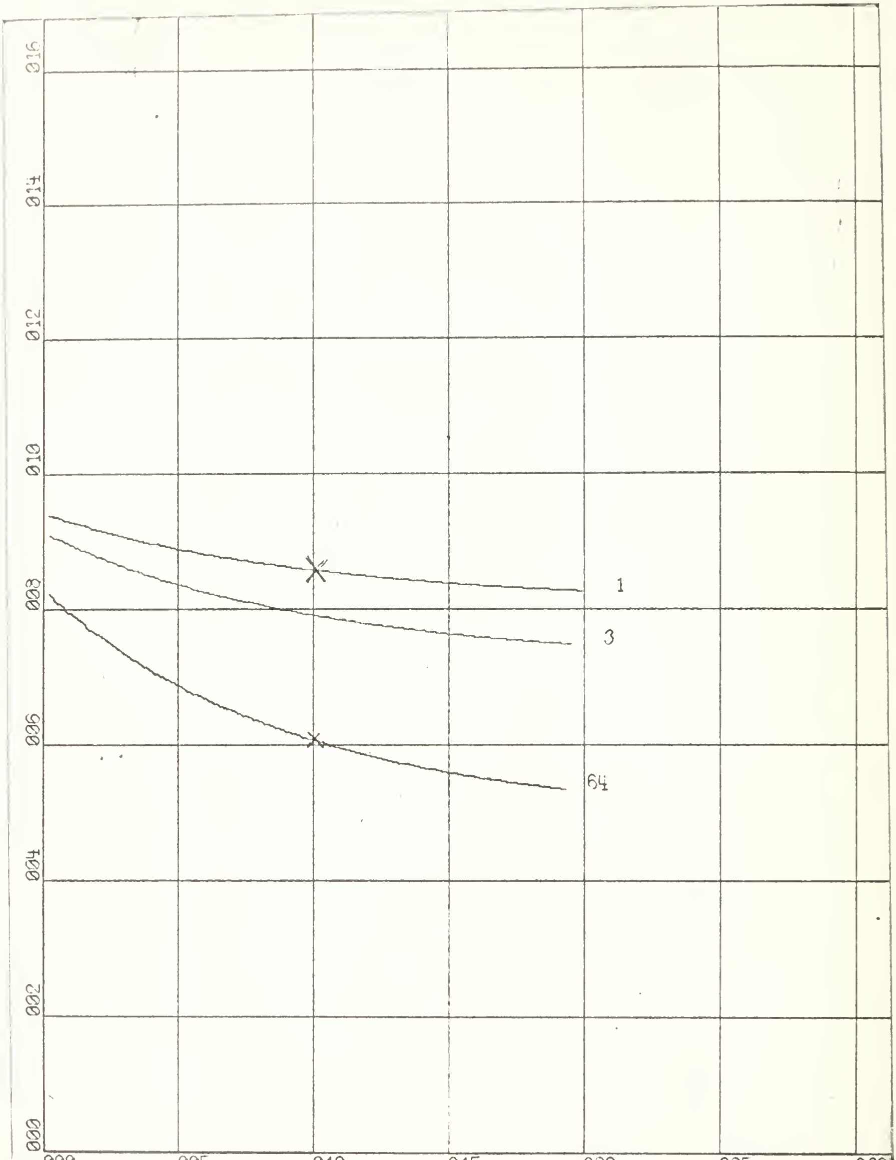
These figures are obtained using sample plan four.

Sample Plan No.	Year 1	Year 2	Year 3	Year 4	Year 5	Year 6	Year 7	Year 8	Year 9	Year 10
1	.1N	.1N	.1N	.1N	.1N	.1N	.1N	.1N	.1N	.1N
2	.05N	.05N	.05N	.05N	.05N	.15N	.15N	.15N	.15N	.15N
3	.15N	.15N	.15N	.15N	.15N	.05N	.05N	.05N	.05N	.05N
4	.3N	.1N	.1N	.1N	.1N	.1N	.05N	.05N	.05N	.05N
5	.05N	.05N	.05N	.05N	.3N	.1N	.1N	.1N	.1N	.1N
6	.3N	.2N	.1N	.1N	.1N	.1N	.03N	.03N	.02N	.02N

(N = the total number of missiles tested in the first ten years of operation.)

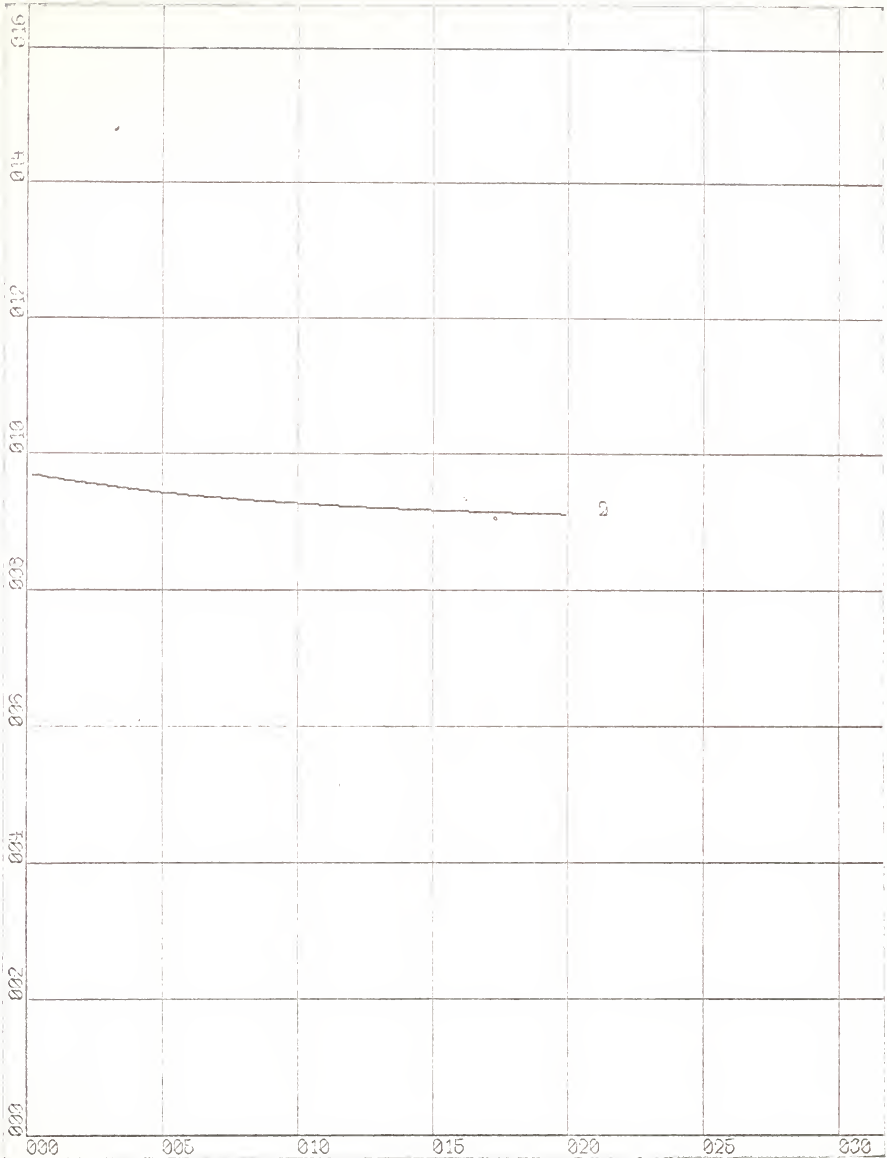
YEARLY TRIALS IN SAMPLE PLANS ONE THROUGH SIX

FIGURE 2. 3. 1



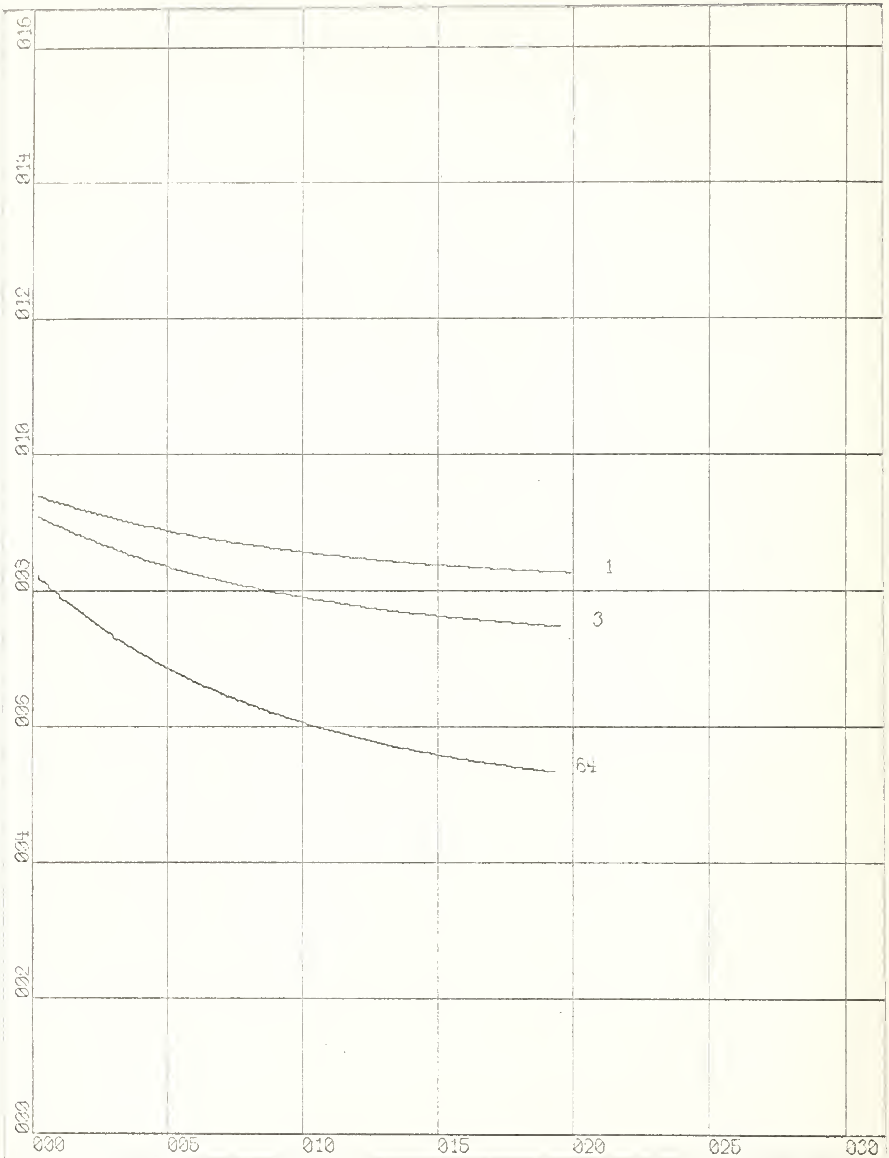
X-SCALE = 5.00E - 01 Units/Inch. P(0) = 7.00E - 01
Y-SCALE = 2.00E - 01 Units/Inch. N = 2.00E + 02

SUM OF VARIANCES VERSUS b



X-SCALE = 5.00E - 01 Units/Inch. P(0) = 7.00E - 01
Y-SCALE = 2.00E - 01 Units/Inch. N = 2.00E + 02

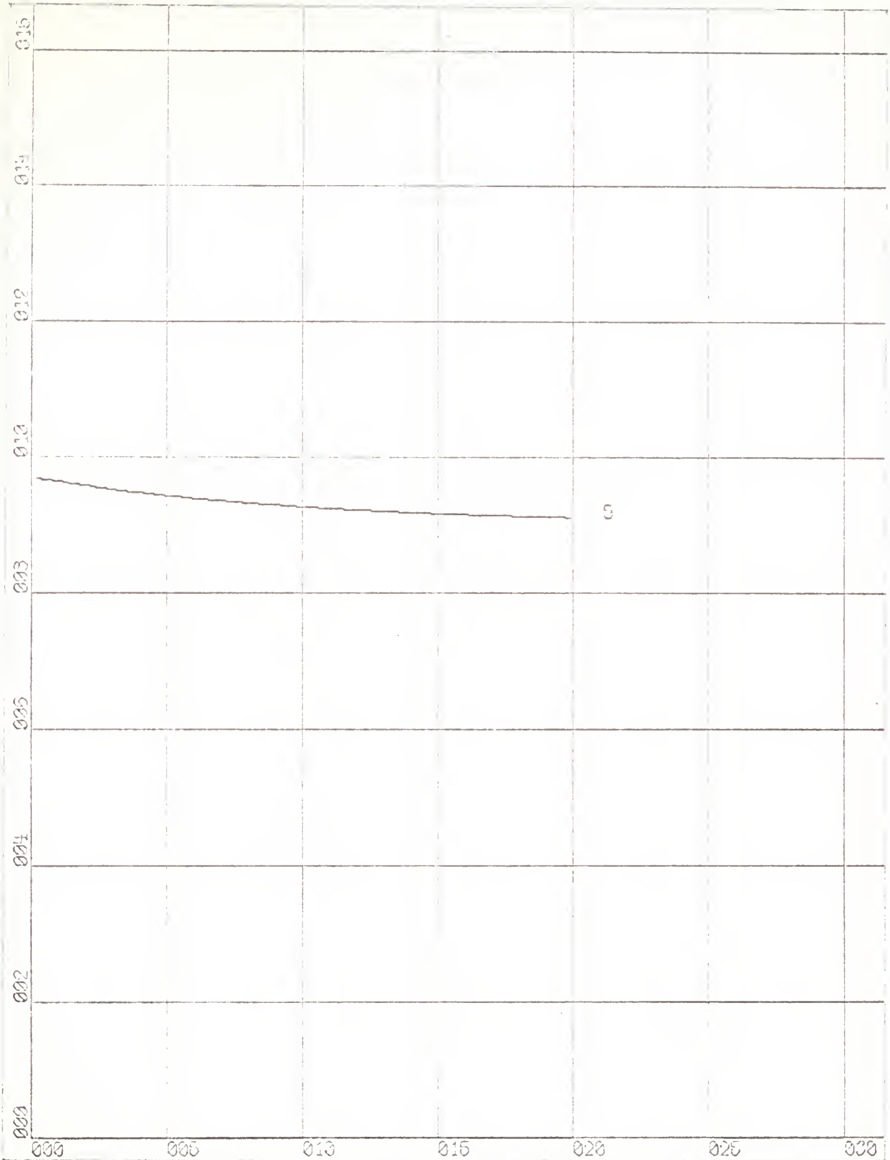
SUM OF VARIANCES VERSUS b



X-SCALE = 5.00E - 01 Units/Inch.
Y-SCALE = 2.00E - 01 Units/Inch.

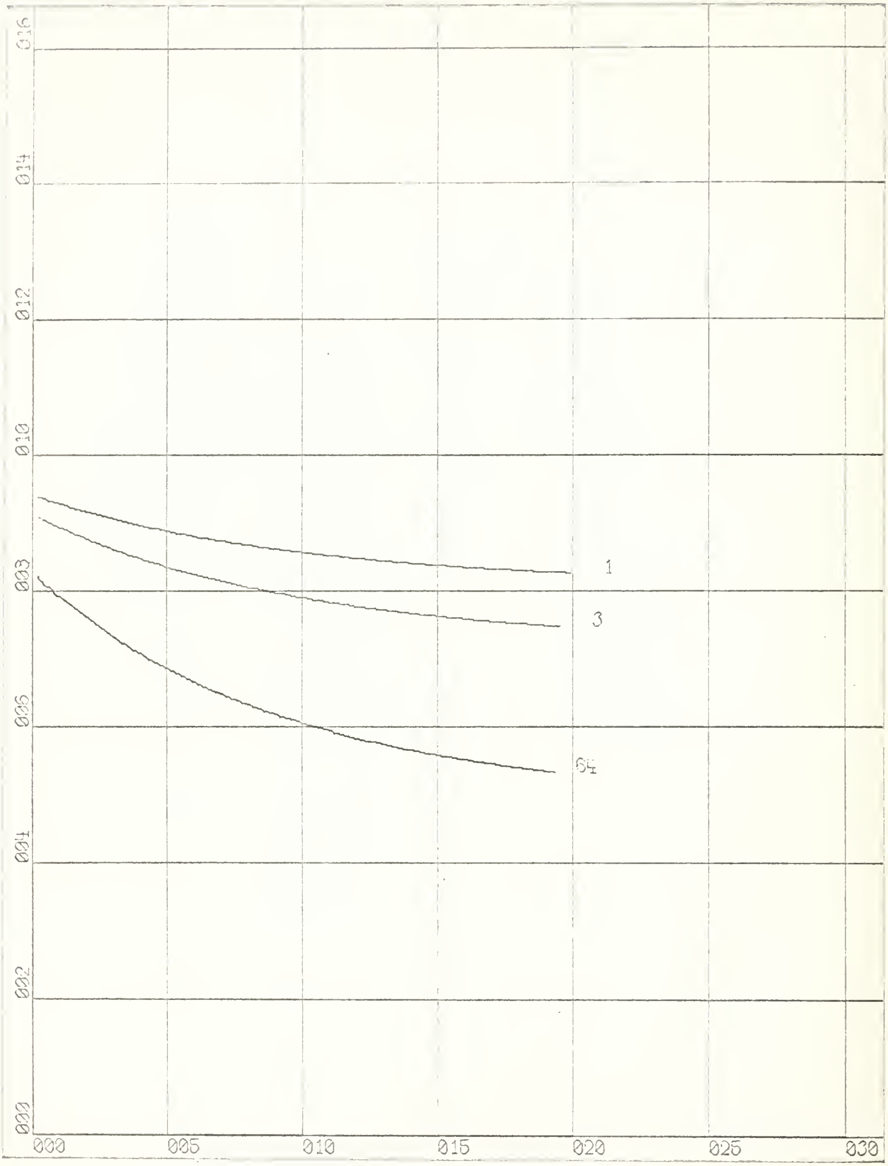
P(0) = 7.00E - 01
N = 3.00E + 02

SUM OF VARIANCES VERSUS b



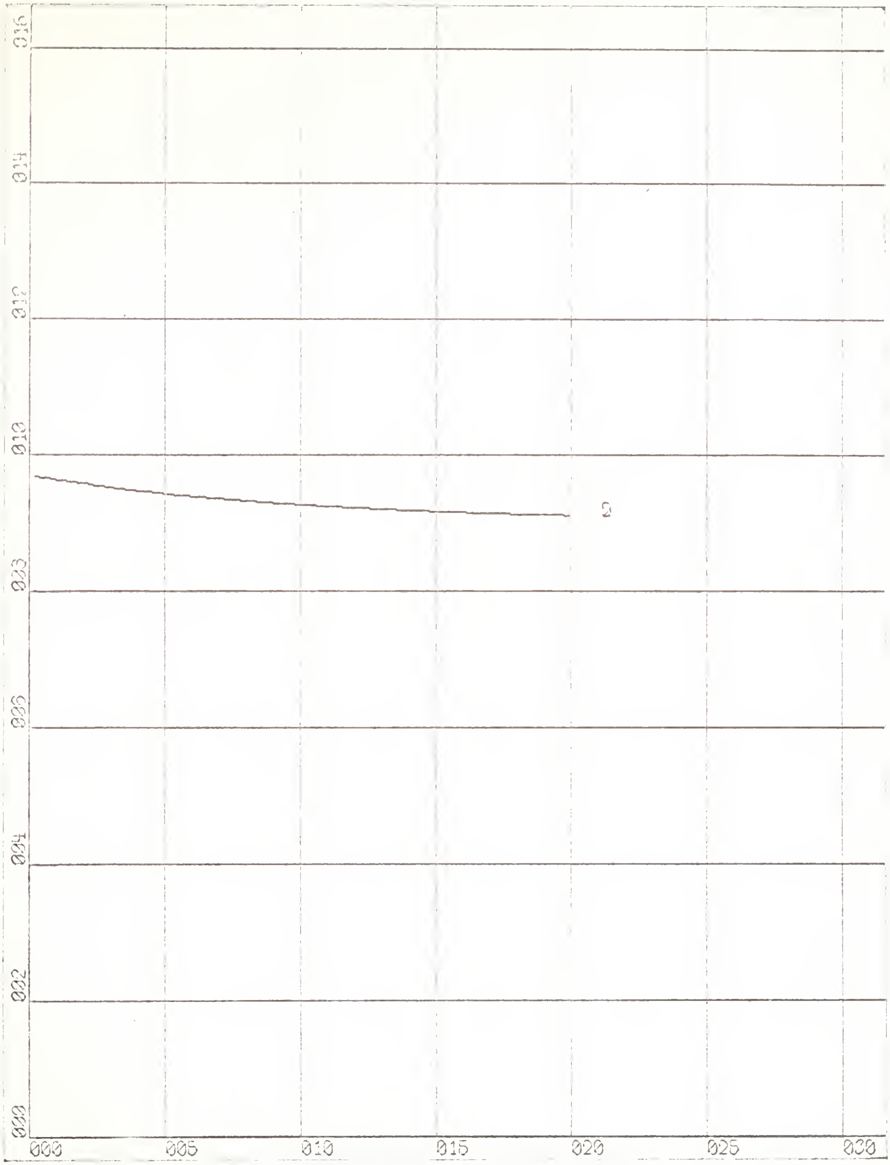
X-SCALE = 5.00E - 01 Units/Inch. P(0) = 7.00E - 01
Y-SCALE = 2.00E - 01 Units/Inch. N = 3.00E + 02

SUM OF VARIANCES VERSUS b



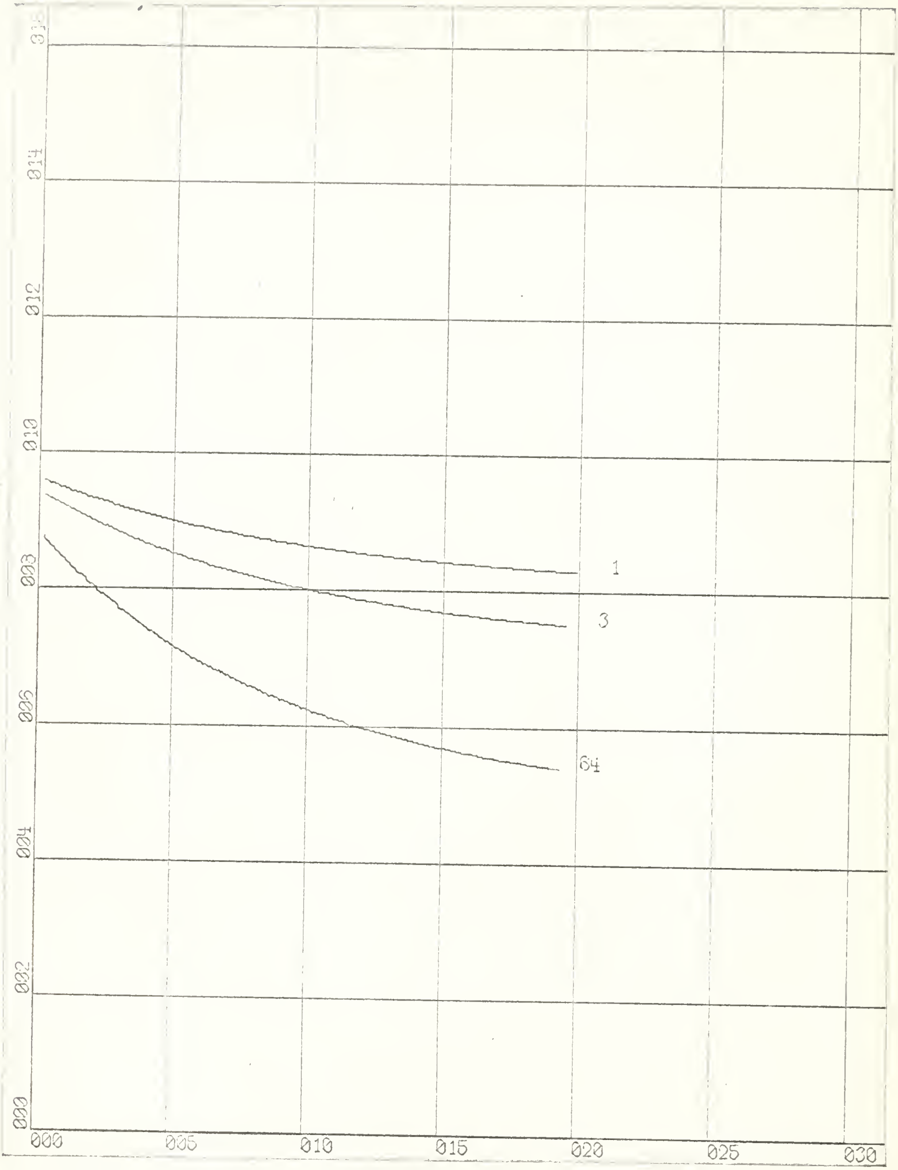
X-SCALE = 5.00E - 01 Units/Inch. P(0) = 7.00E - 01
Y-SCALE = 2.00E - 01 Units/Inch. N = 4.00E + 02

SUM OF VARIANCES VERSUS b



X-SCALE = 5.00E - 01 Units/Inch. P(0) = 7.00E - 01
Y-SCALE = 2.00E - 01 Units/Inch. N = 4.00E + 02

SUM OF VARIANCES VERSUS b



X-SCALE = 5.00E - 01 Units/Inch.
Y-SCALE = 2.00E - 01 Units/Inch.

P(0) = 8.00E - 01
N = 2.00E + 02

SUM OF VARIANCES VERSUS b

X-SCALE = 5.00E - 01 Units/Inch.
Y-SCALE = 2.00E - 01 Units/Inch.

P(0) = 8.00E - 01
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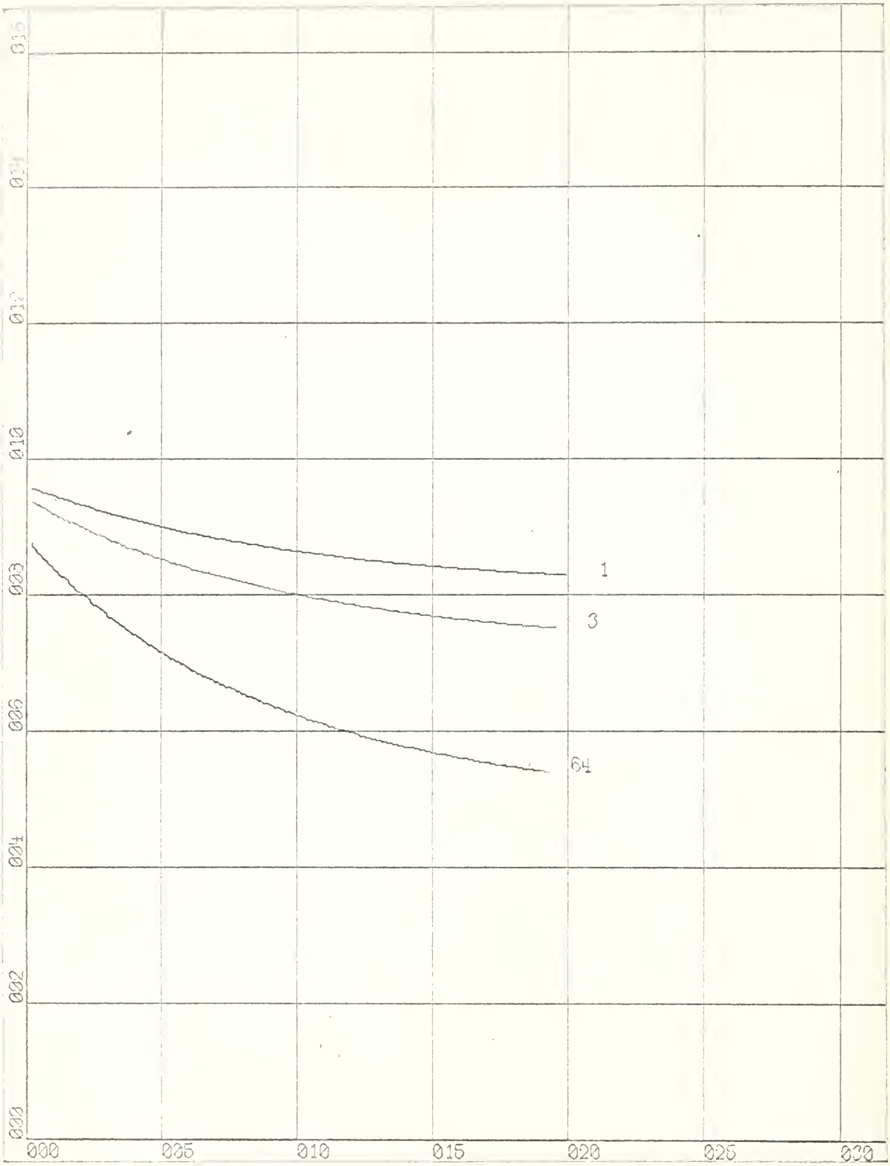
SUM OF VARIANCES VERSUS b



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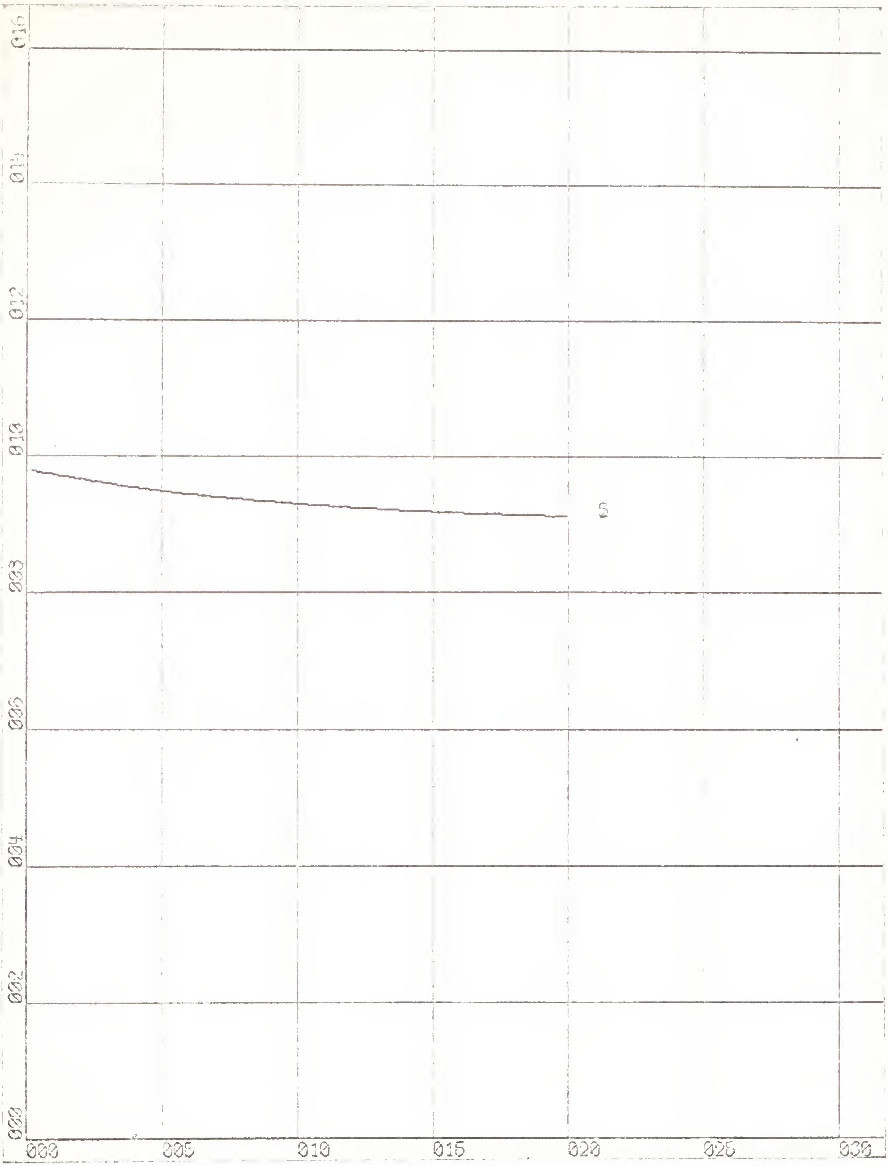
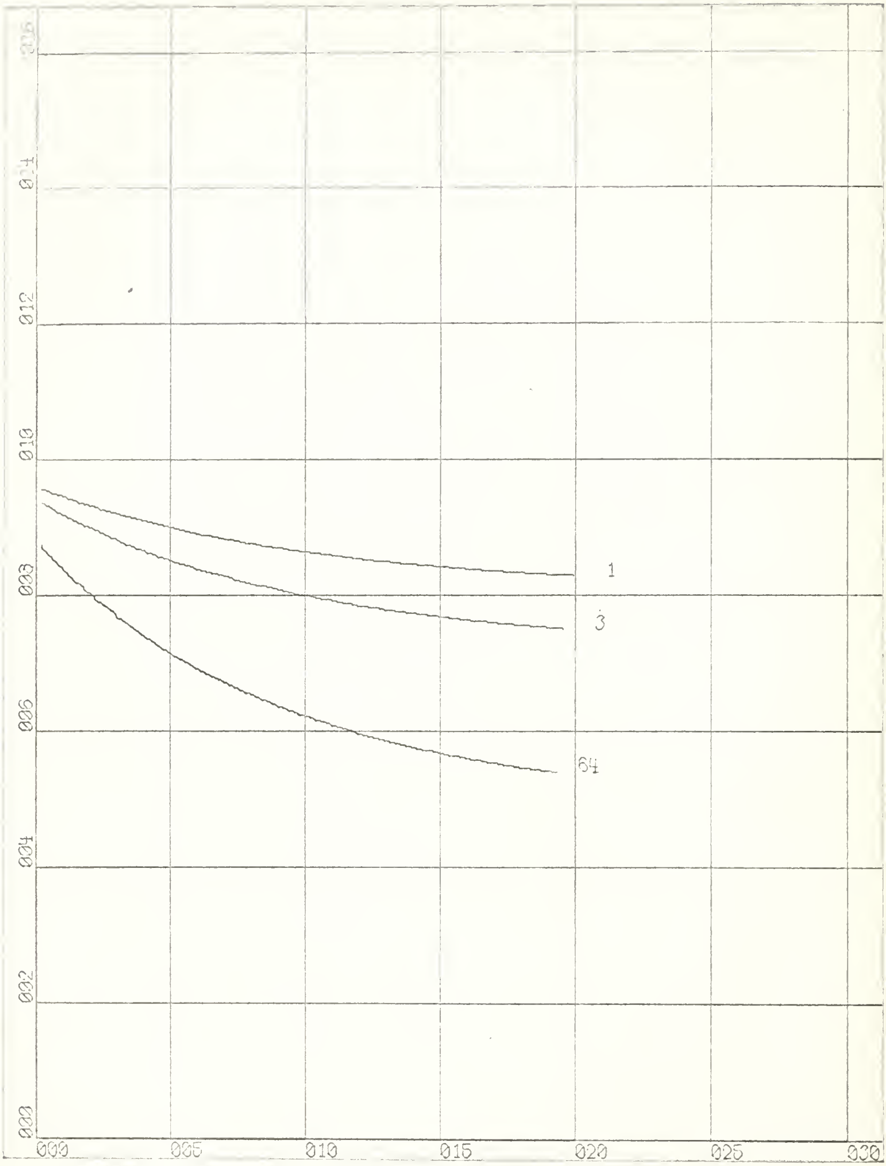
Y-SCALE = 2.00E - 01 Units/Inch. N = 3.00E + 02

SUM OF VARIANCES VERSUS b



X-SCALE = 5.00E - 01 Units/Inch. P(0) = 8.00E - 01
Y-SCALE = 2.00E - 01 Units/Inch. N = 3.00E + 02

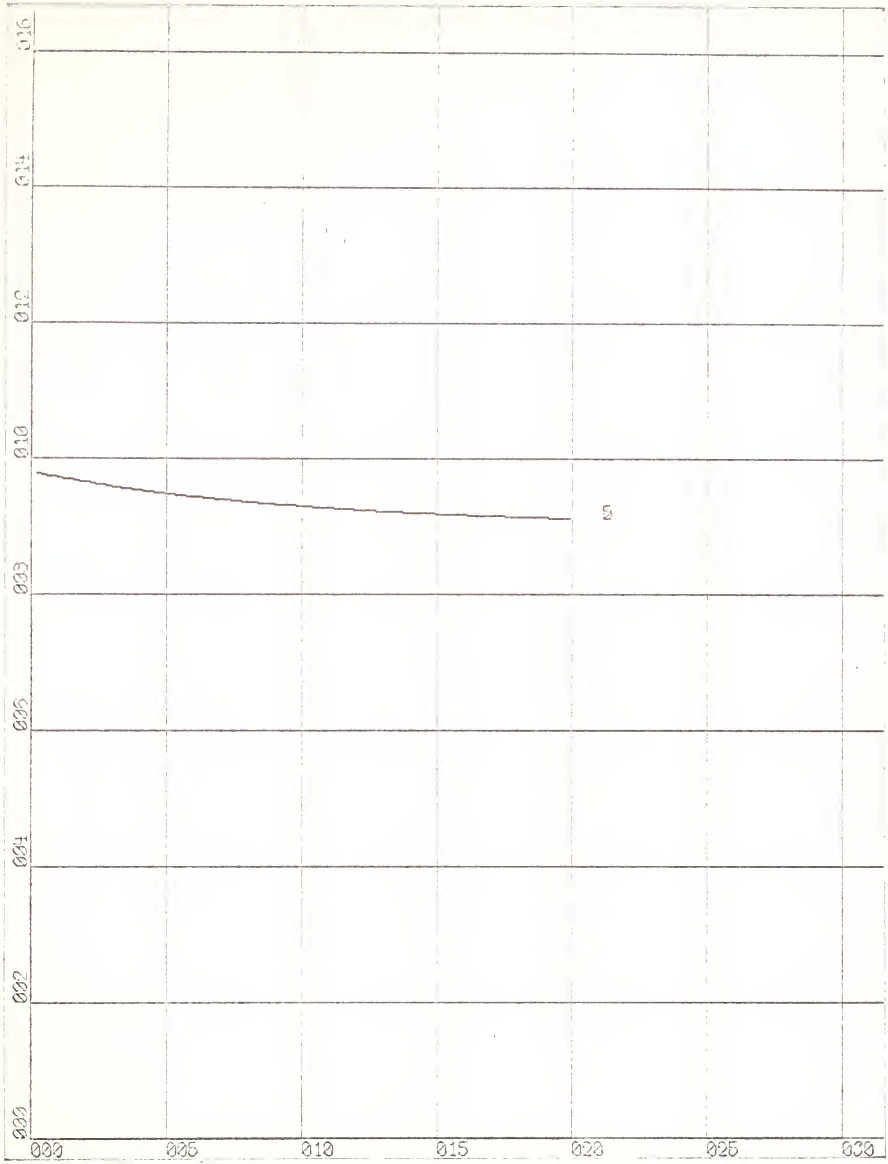
SUM OF VARIANCES VERSUS b



X-SCALE = 5.00E - 01 Units/Inch.
Y-SCALE = 2.00E - 01 Units/Inch.

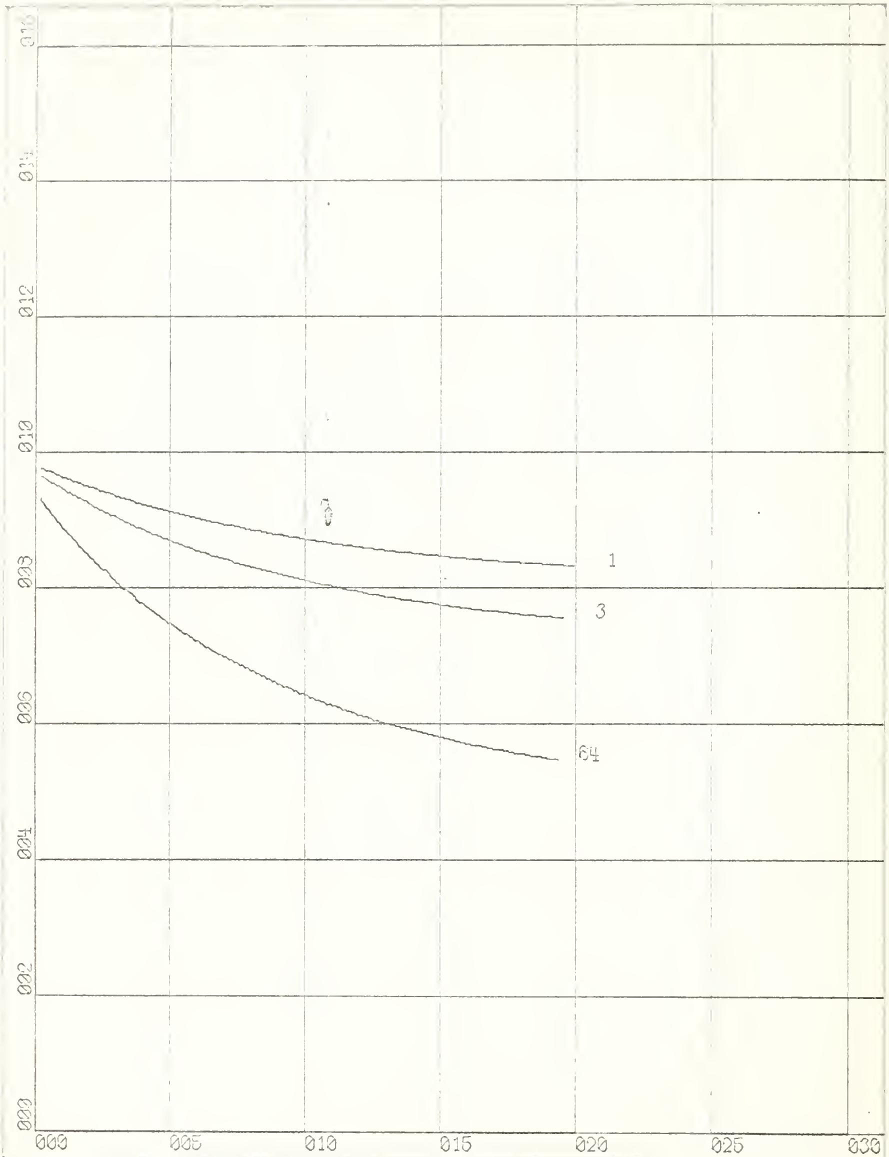
P(0) = 8.00E - 01
N = 4.00E + 02

SUM OF VARIANCES VERSUS b



X-SCALE = 5.00E - 01 Units/Inch. P(0) = 8.00E - 01
Y-SCALE = 2.00E - 01 Units/Inch. N = 4.00E + 02

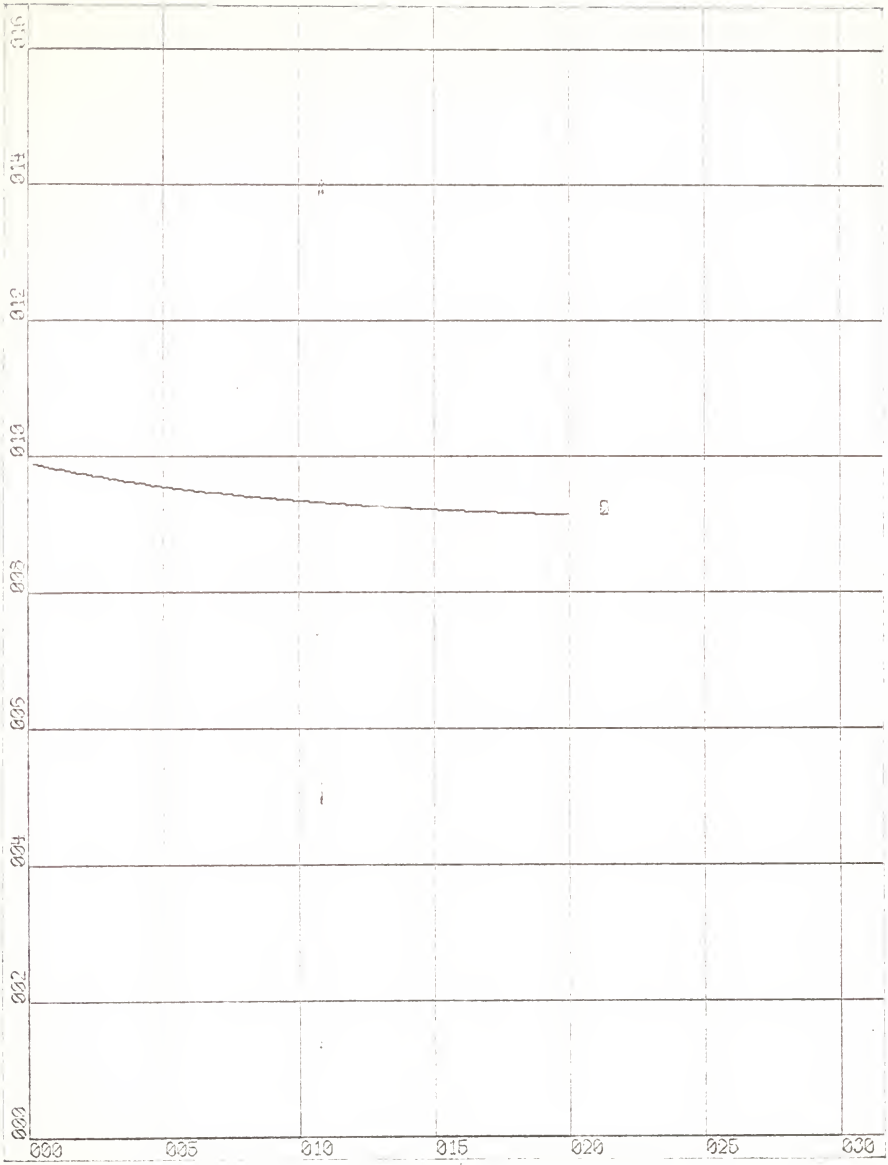
SUM OF VARIANCES VERSUS b



X-SCALE = 5.00E - 01 Units/Inch.
Y-SCALE = 2.00E - 01 Units/Inch.

P(0) = 9.00E - 01
N = 2.00E + 02

SUM OF VARIANCES VERSUS b



X-SCALE = 5.00E - 01 Units/Inch. P(0) = 9.00E - 01
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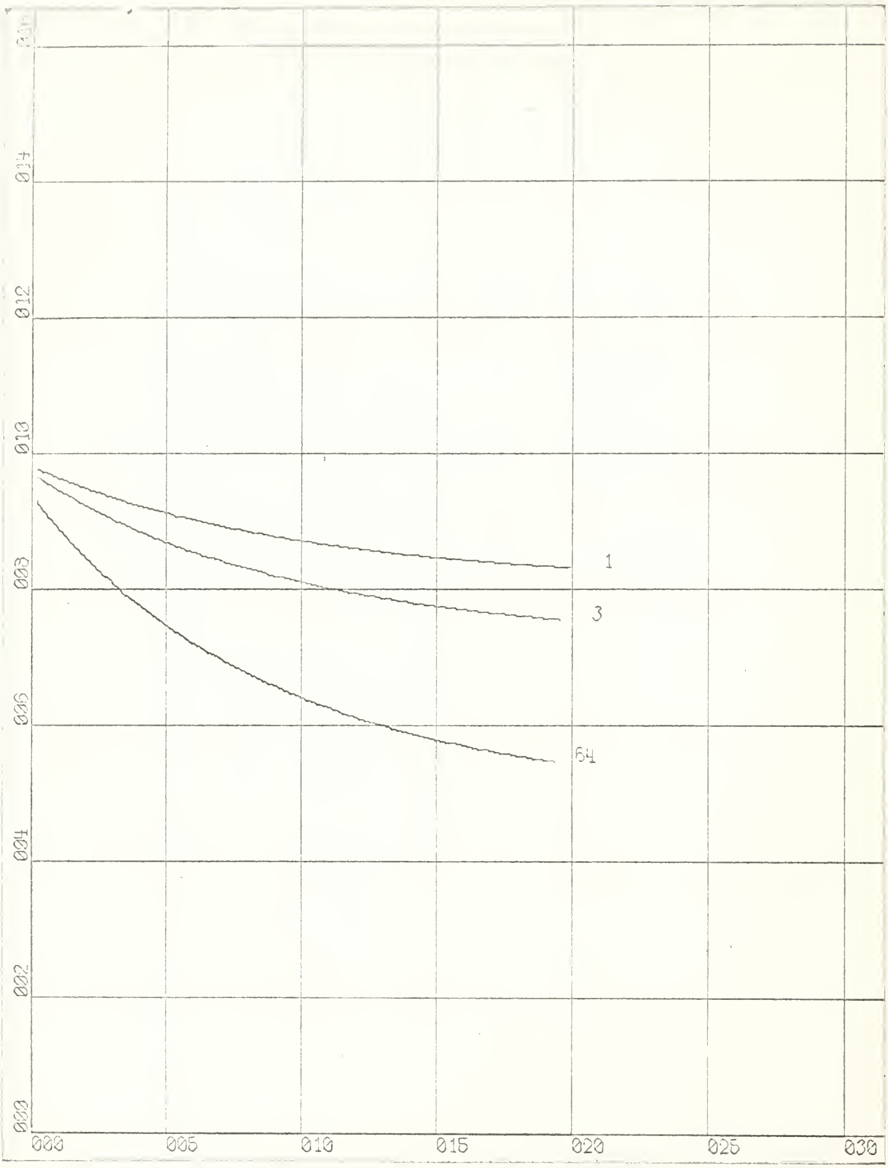
SUM OF VARIANCES VERSUS b



X-SCALE = 5.00E - 01 Units/Inch.
Y-SCALE = 2.00E - 01 Units/Inch.

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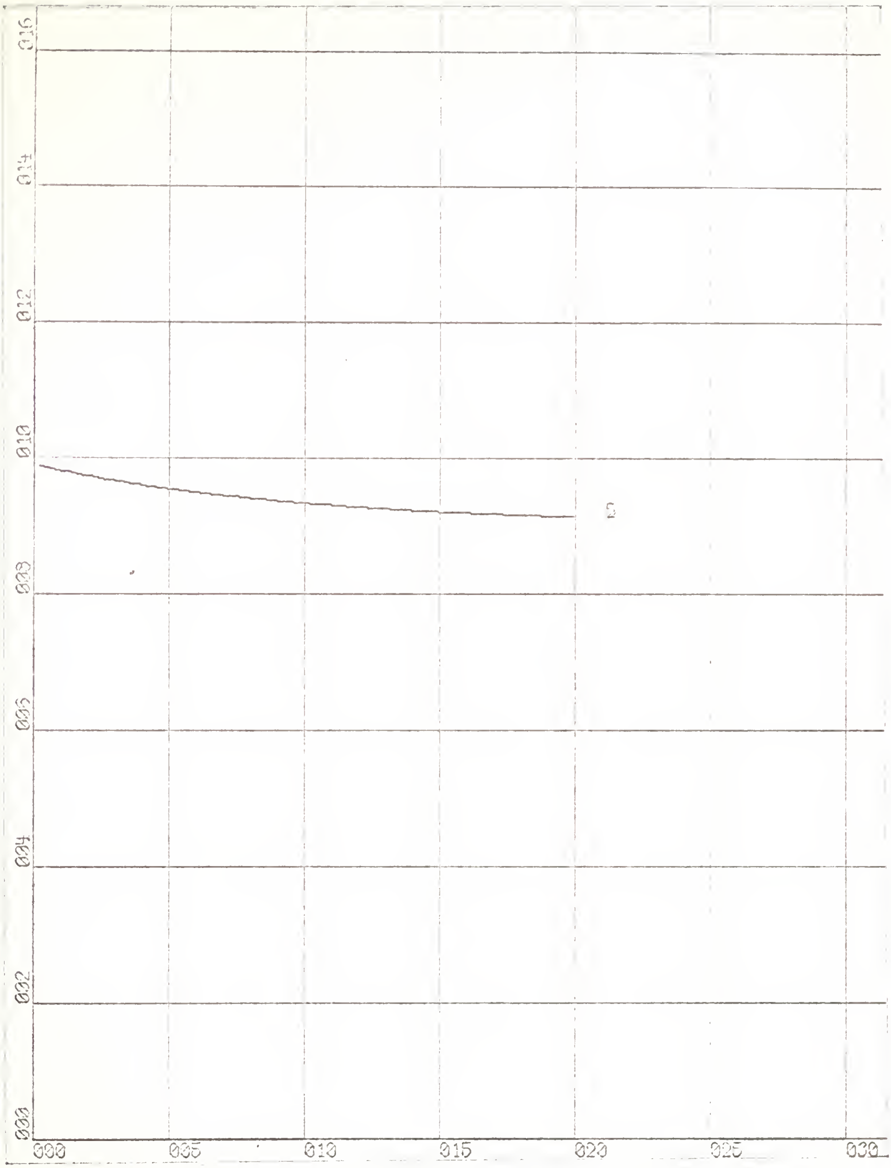
N = 3.00E + 02

SUM OF VARIANCES VERSUS b



X-SCALE = 5.00E - 01 Units/Inch.
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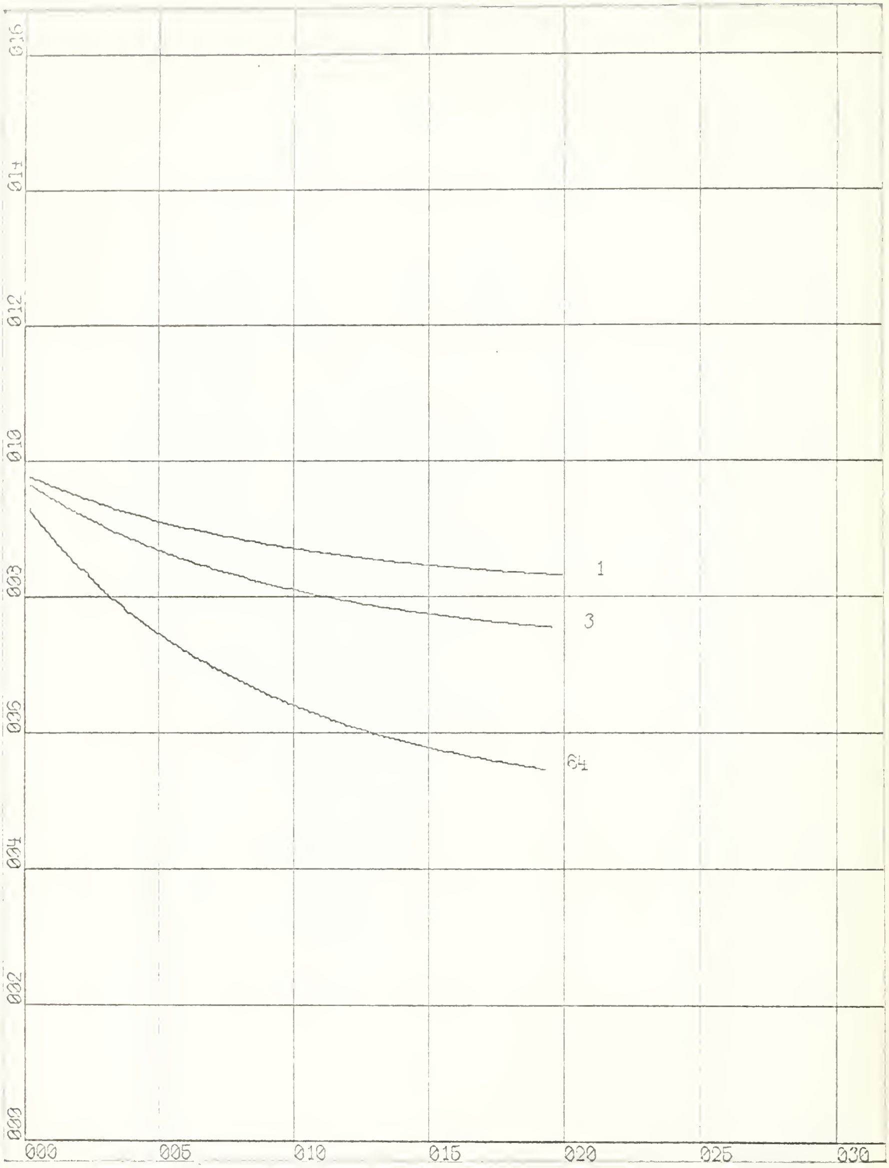
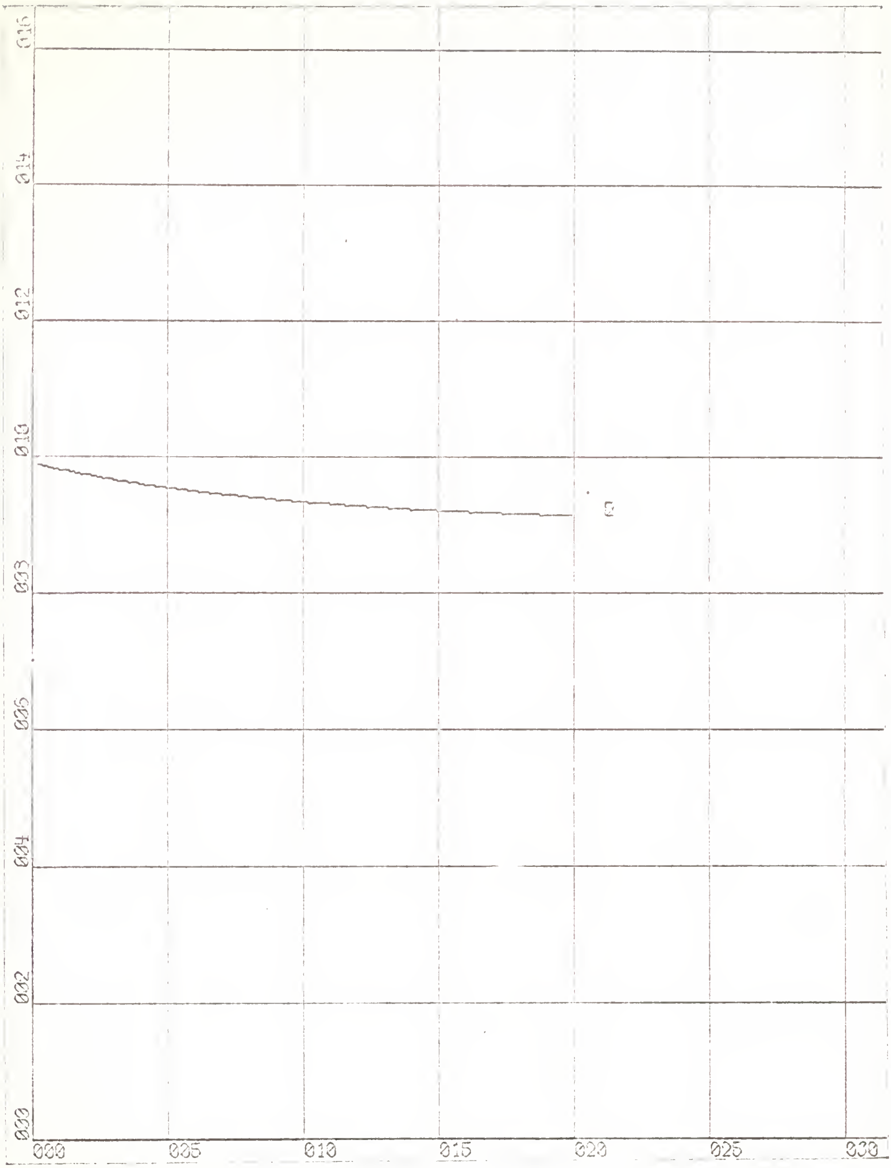
P(0) = 9.00E - 01
N = 3.00E + 02

SUM OF VARIANCES VERSUS b



X-SCALE = 5.00E - 01 Units/Inch. P(0) = 9.00E - 01
Y-SCALE = 2.00E - 01 Units/Inch. N = 4.00E + 02

SUM OF VARIANCES VERSUS b



X-SCALE = 5.00E - 01 Units/Inch. P(0) = 9.00E - 01
Y-SCALE = 2.00E - 01 Units/Inch. N = 4.00E + 02

SUM OF VARIANCES VERSUS b

2.4 Determining FOT Test Sizes Using Model Two

Model two uses the following expression for reliability at any time t:

$$R(t) = p_0 - Be^{ct} \qquad t > 0$$

where:

p_0 = initial reliability;

B = a known constant;

c = an unknown parameter.

Notice that this expression assumes an exponential decay of reliability as time increases. The factor B is chosen so that R(t) is always non-negative. The graph of e^x for positive x starts at a value of one and is monotonically increasing as x increases, see figure 2.4.2. If a scaling factor was not used in the expression for reliability, then negative values would be obtained. Arbitrarily, a scaling factor of B = .01 was used. This merely displaces the curve of the second term in the expression downward, see figure 2.4.2.

To investigate Model Two, a Maximum Likelihood Estimator for reliability was obtained. Then, the expression for the variance of this estimator was determined, see section 3.4. The sum of the variances was then computed for all combinations of p_0 and N, as explained in section 2.2. In addition, these values of the sum of the variances were computed for values of c ranging from .00424 to .424. Note that if

larger values of c were allowed, R(t) would be negative when t = 10. Therefore, c has been varied over its entire range of plausible values.

The results of the computations of the sum of the variances are displayed on the graphs on pages 42 through 59. The abscissa of these graphs represents values of c. The ordinate of these graphs represents values of the sum of the variances. See page 17 for an explanation of how to read these graphs.

By examining these graphs, it can be seen that the sample plan that gives the minimum sum of the variances for all combinations of p_0 and N is sample plan six. The value of the sum of the variances then increases for the sample plans in the following order: plan four, plan three, plan one, plan five, and then plan two.

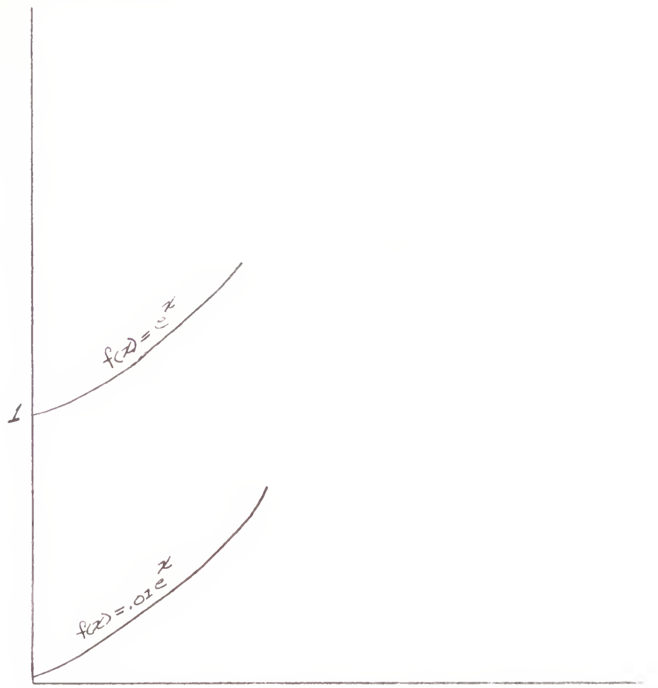
For convenience in understanding these results, the table explaining the yearly test sizes for each sample plan is reproduced on the following page. Notice that as in the first model the sum of the variances is less for the plans that call for heavy testing in the early years. The plans that call for heavy testing in the out years result in the largest values of the sum of the variances.

Sample Plan No.	Year 1	Year 2	Year 3	Year 4	Year 5	Year 6	Year 7	Year 8	Year 9	Year 10
1	.1N	.1N	.1N	.1N	.1N	.1N	.1N	.1N	.1N	.1N
2	.05N	.05N	.05N	.05N	.05N	.15N	.15N	.15N	.15N	.15N
3	.15N	.15N	.15N	.15N	.15N	.05N	.05N	.05N	.05N	.05N
4	.3N	.1N	.1N	.1N	.1N	.1N	.05N	.05N	.05N	.05N
5	.05N	.05N	.05N	.05N	.3N	.1N	.1N	.1N	.1N	.1N
6	.3N	.2N	.1N	.1N	.1N	.1N	.03N	.03N	.02N	.02N

(N = the total number of missiles tested in the first ten years of operation.)

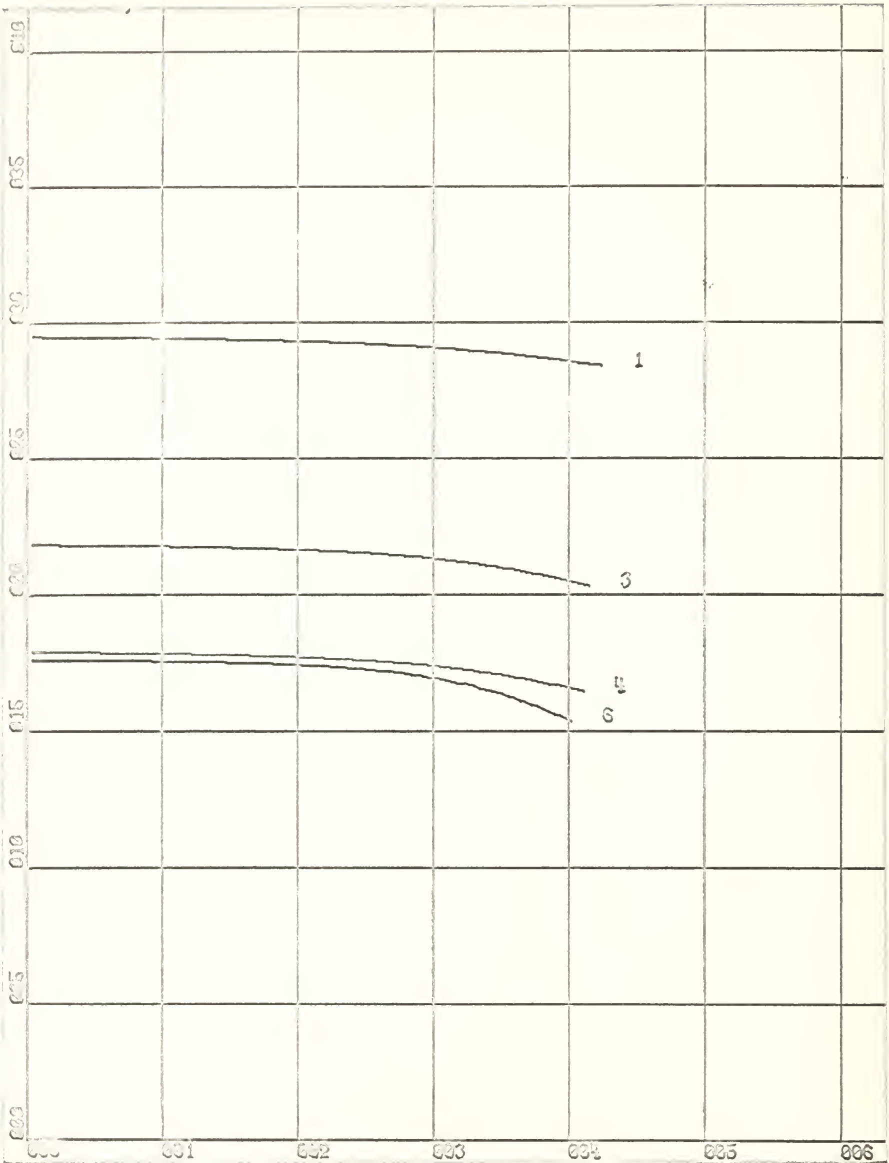
YEARLY TRIALS IN SAMPLE PLANS ONE THROUGH SIX

FIGURE 2. 4. 1



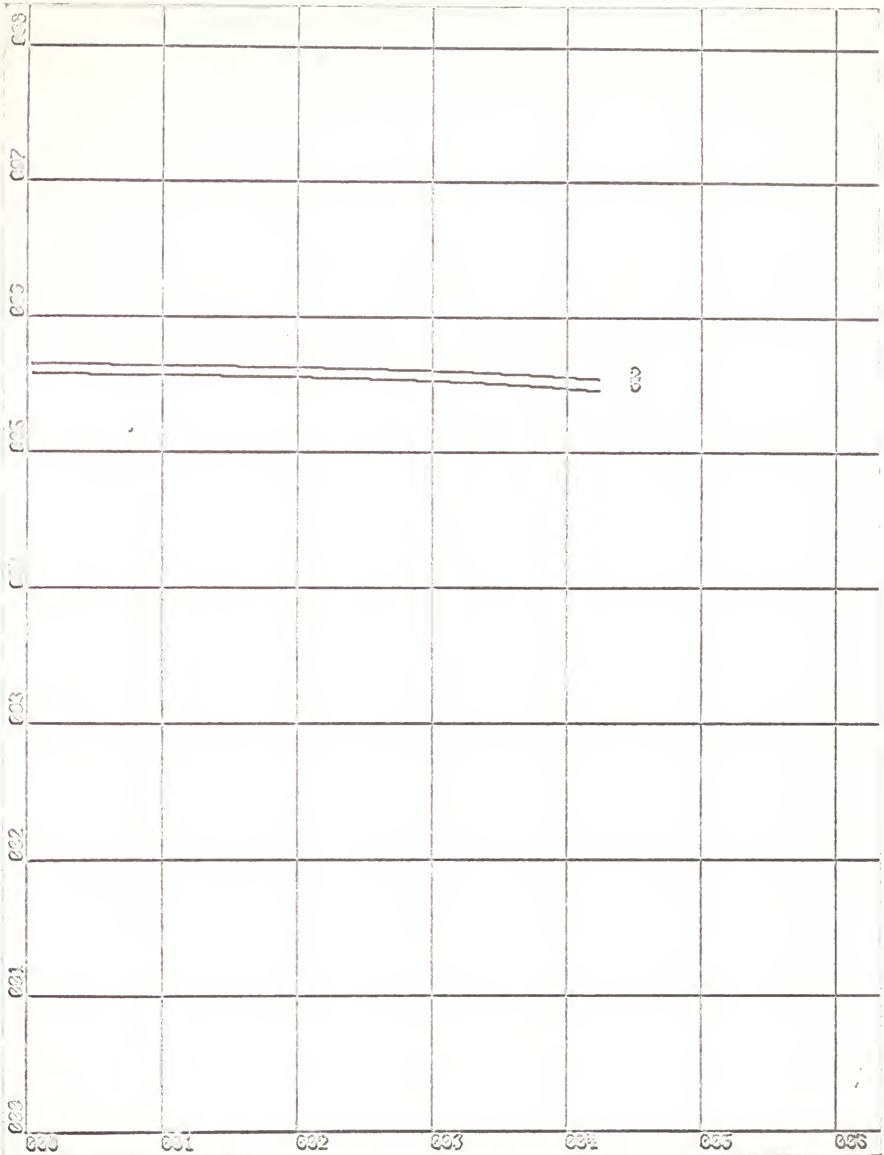
GRAPH OF e^x VERSUS x

FIGURE 2.4.2



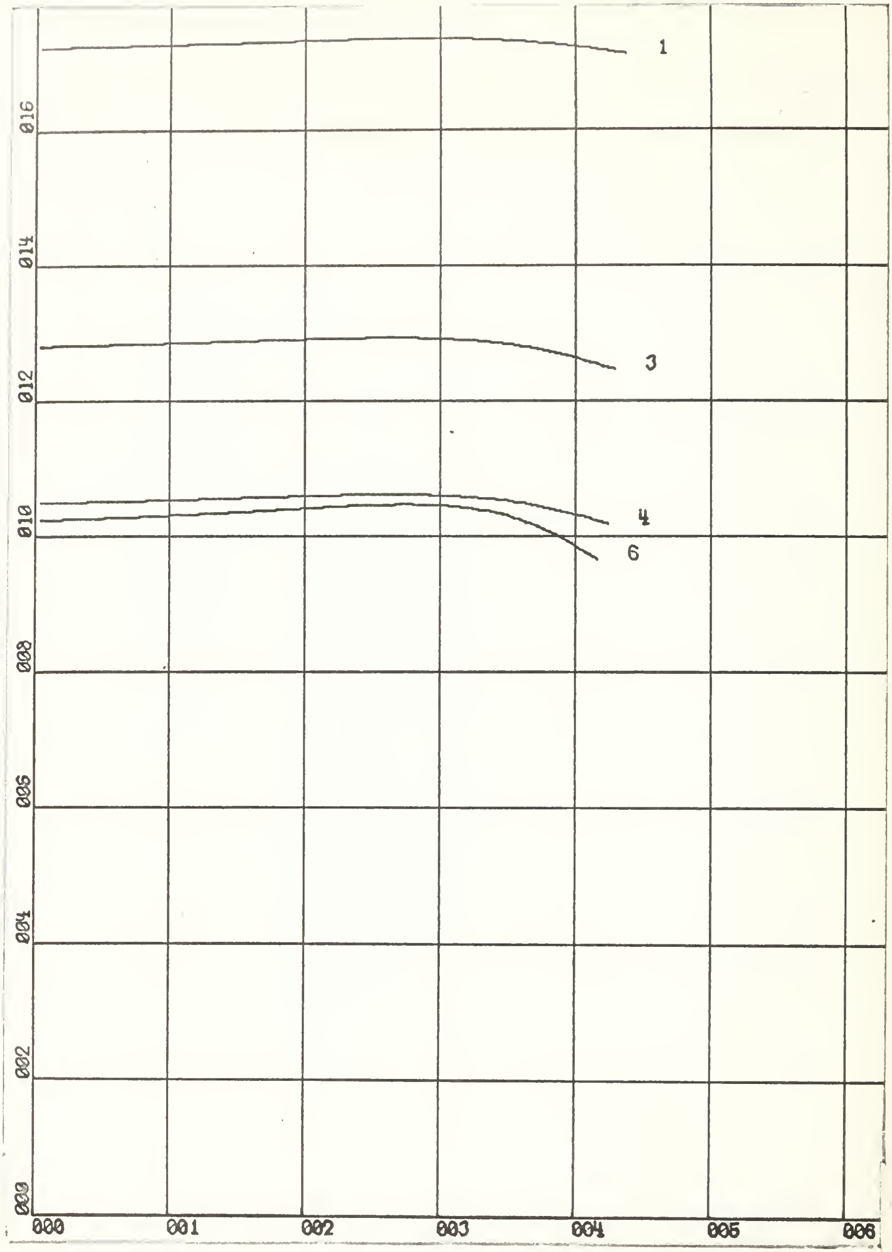
X-SCALE = 5.00E - 01 Units/Inch. P(0) = 7.00E - 01
Y-SCALE = 5.00E - 02 Units/Inch. N = 2.00E + 02

SUM OF VARIANCES VERSUS c



X-SCALE = 1.00E - 01 Units/Inch. P(0) = 7.00E - 01
Y-SCALE = 1.00E - 01 Units/Inch. N = 2.00E + 02

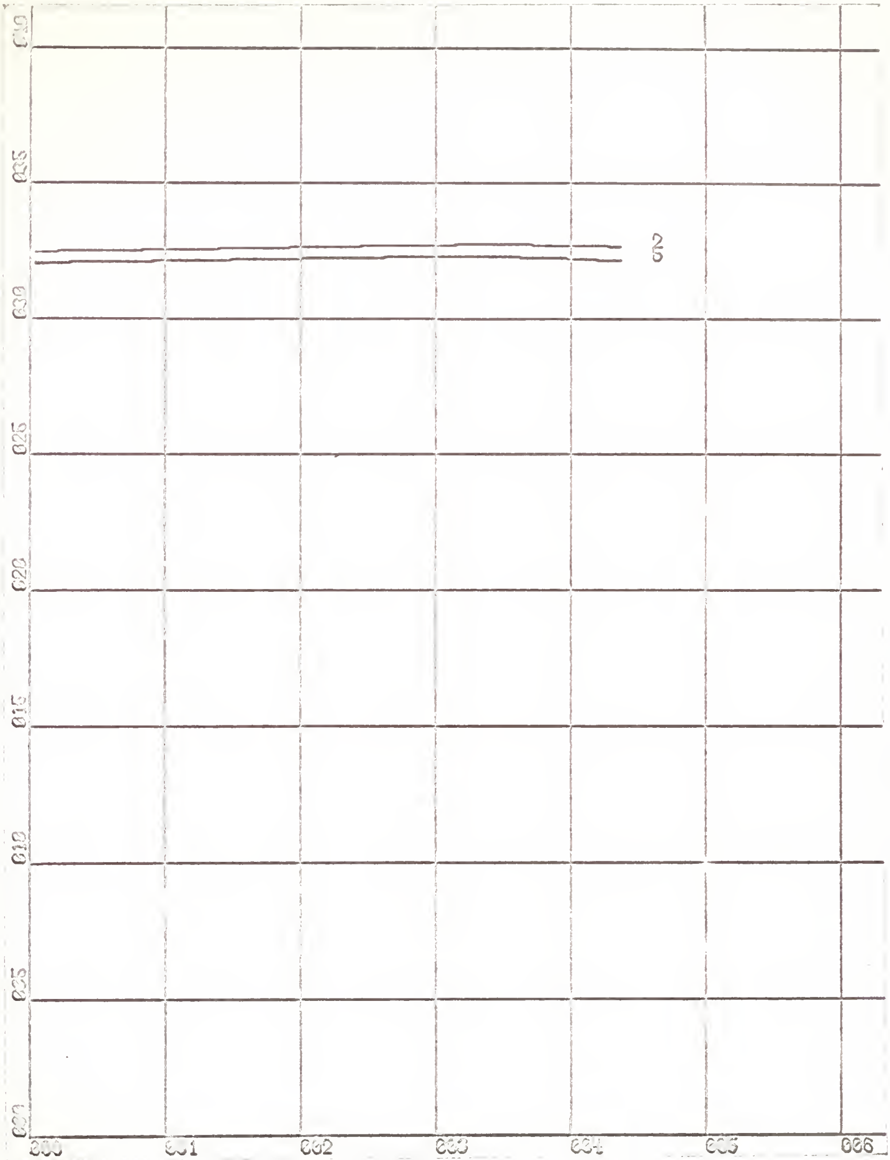
SUM OF VARIANCES VERSUS c



X-SCALE = 1.00E - 01 Units/Inch.
Y-SCALE = 2.00E - 02 Units/Inch.

P(0) = 8.00E - 01
N = 2.00E + 02

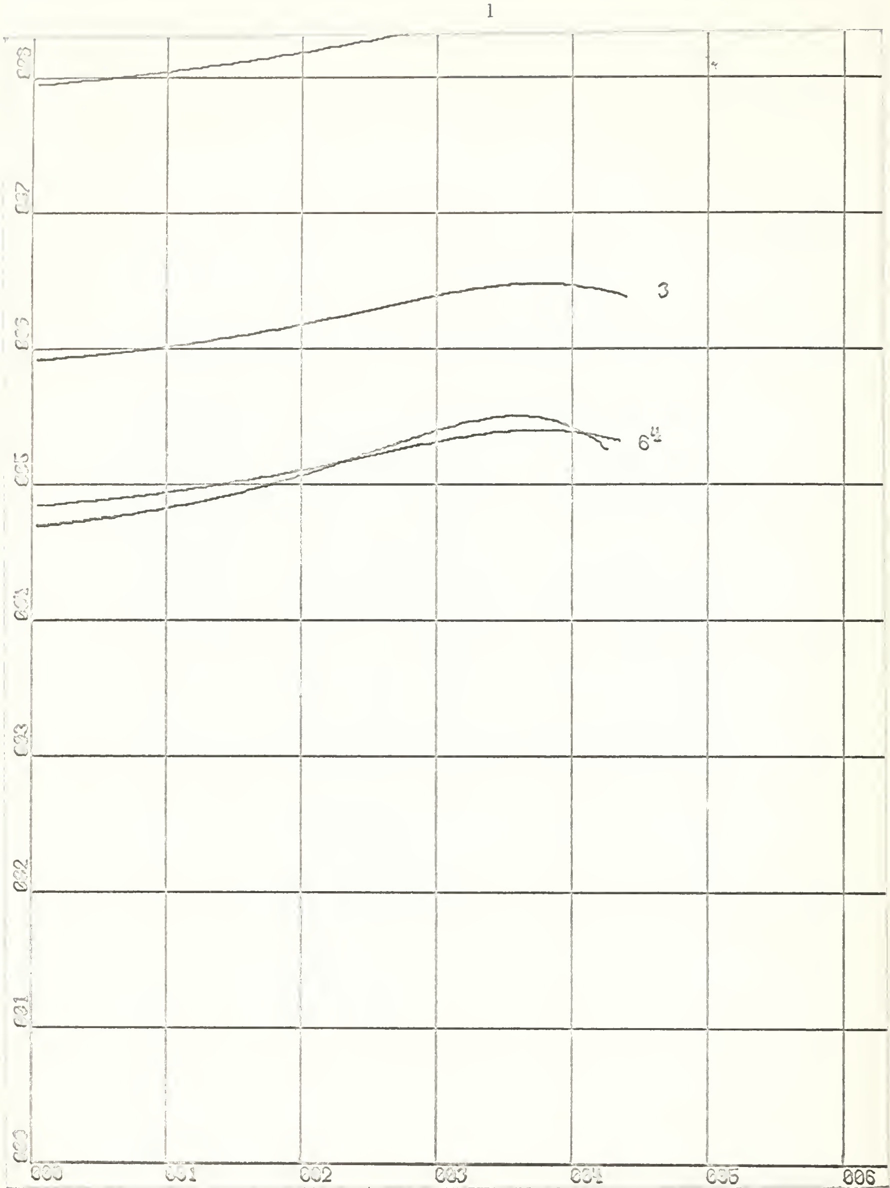
SUM OF VARIANCES VERSUS c



X-SCALE = 1.00E - 01 Units/Inch.
Y-SCALE = 5.00E - 02 Units/Inch.

P(0) = 8.00E - 01
N = 2.00E + 02

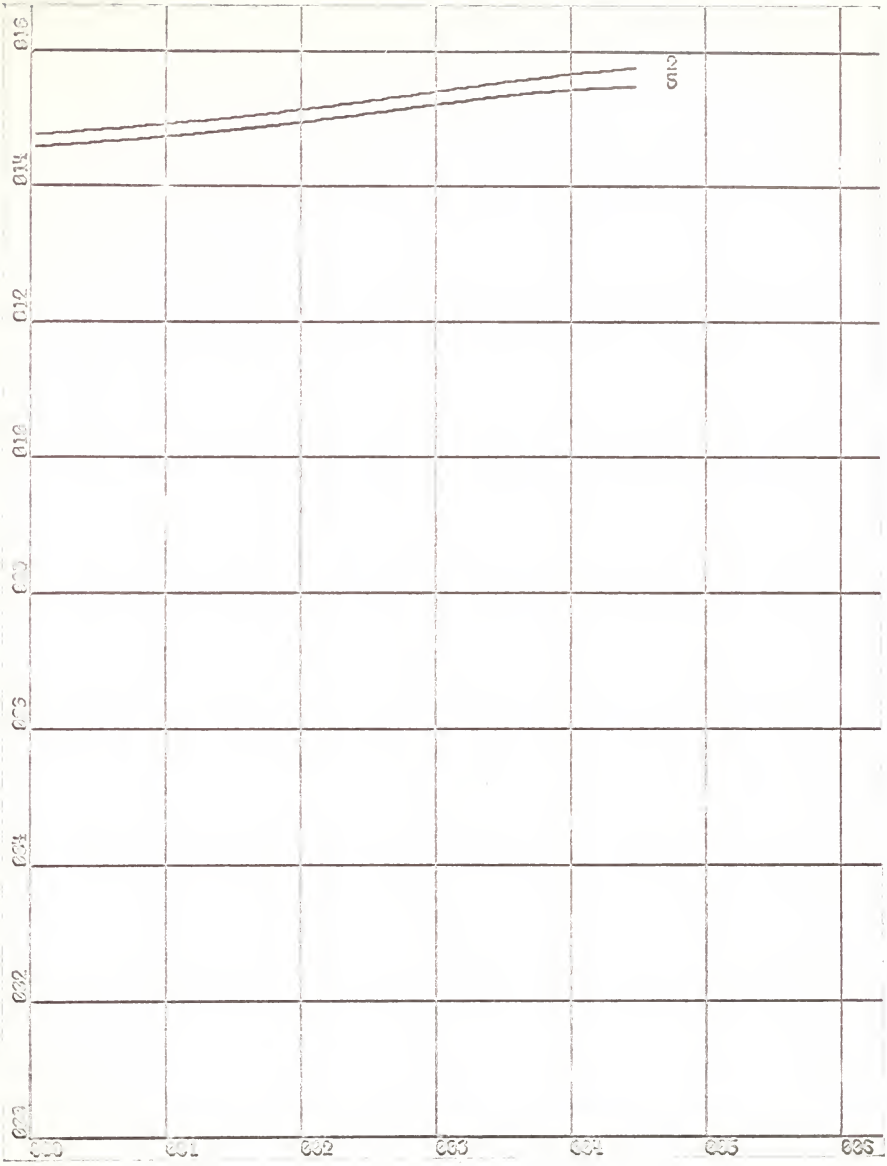
SUM OF VARIANCES VERSUS c



X-SCALE = 1.00E - 01 Units/Inch. P(0) = 9.00E - 01

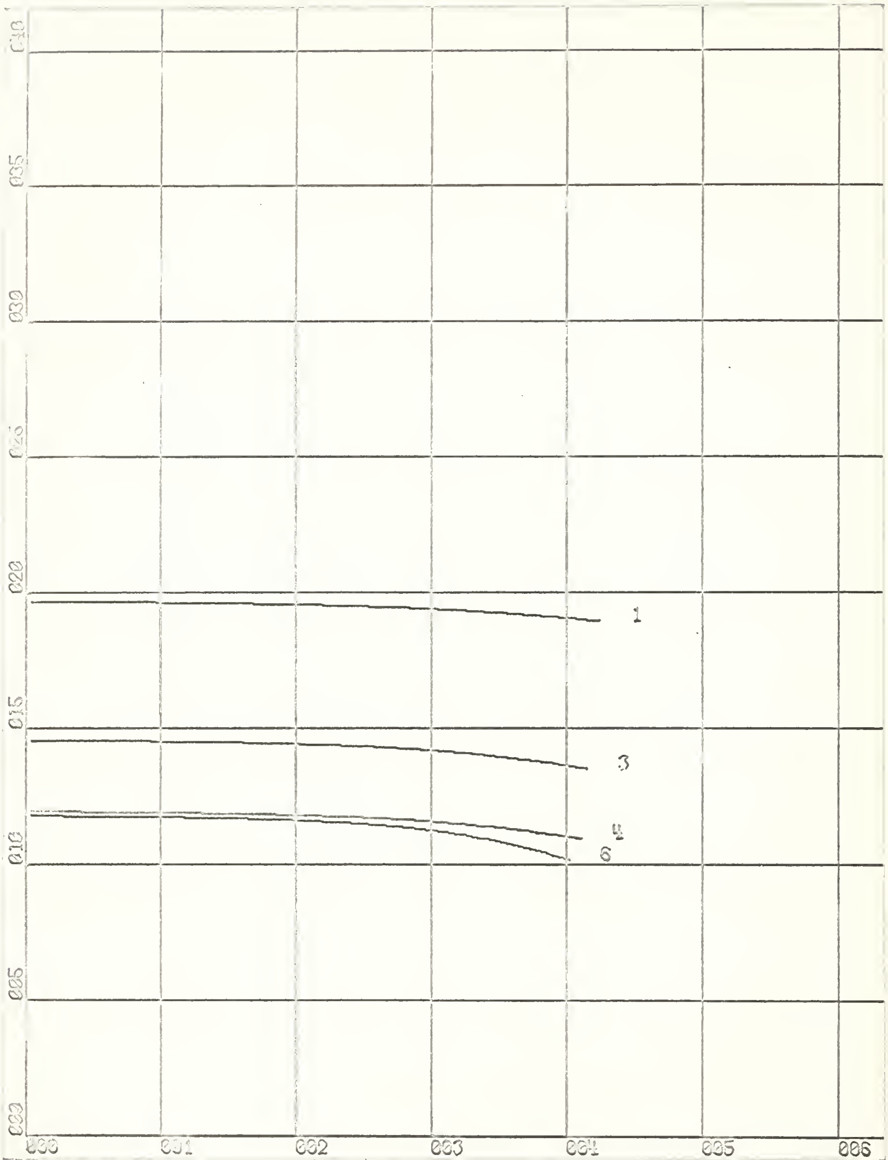
Y-SCALE = 1.00E - 02 Units/Inch. N = 2.00E + 02

SUM OF VARIANCES VERSUS c



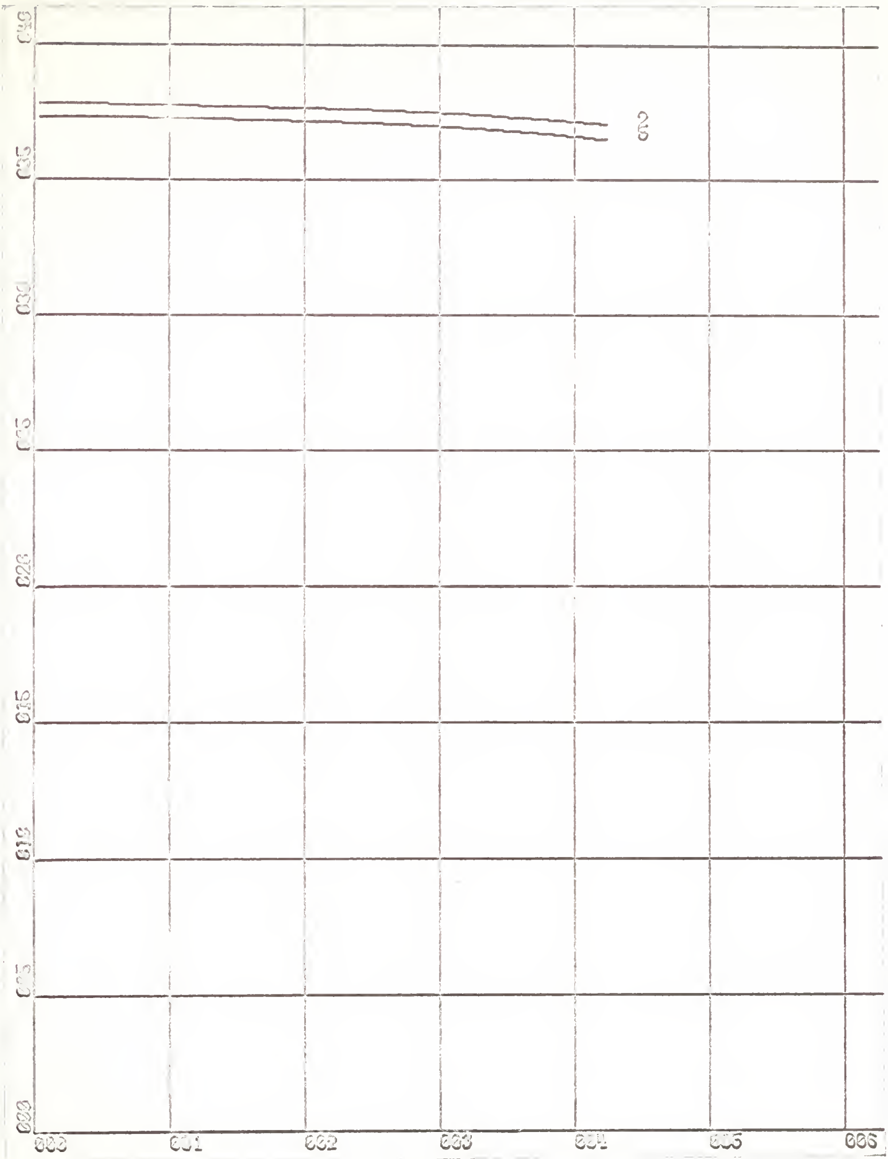
X-SCALE = 1.00E - 01 Units/Inch. P(0) = 9.00E - 01
Y-SCALE = 2.00E - 02 Units/Inch. N = 2.00E + 02

SUM OF VARIANCES VERSUS c



X-SCALE = 1.00E - 01 Units/Inch. P(0) = 7.00E - 01
Y-SCALE = 5.00E - 02 Units/Inch. N = 3.00E + 02

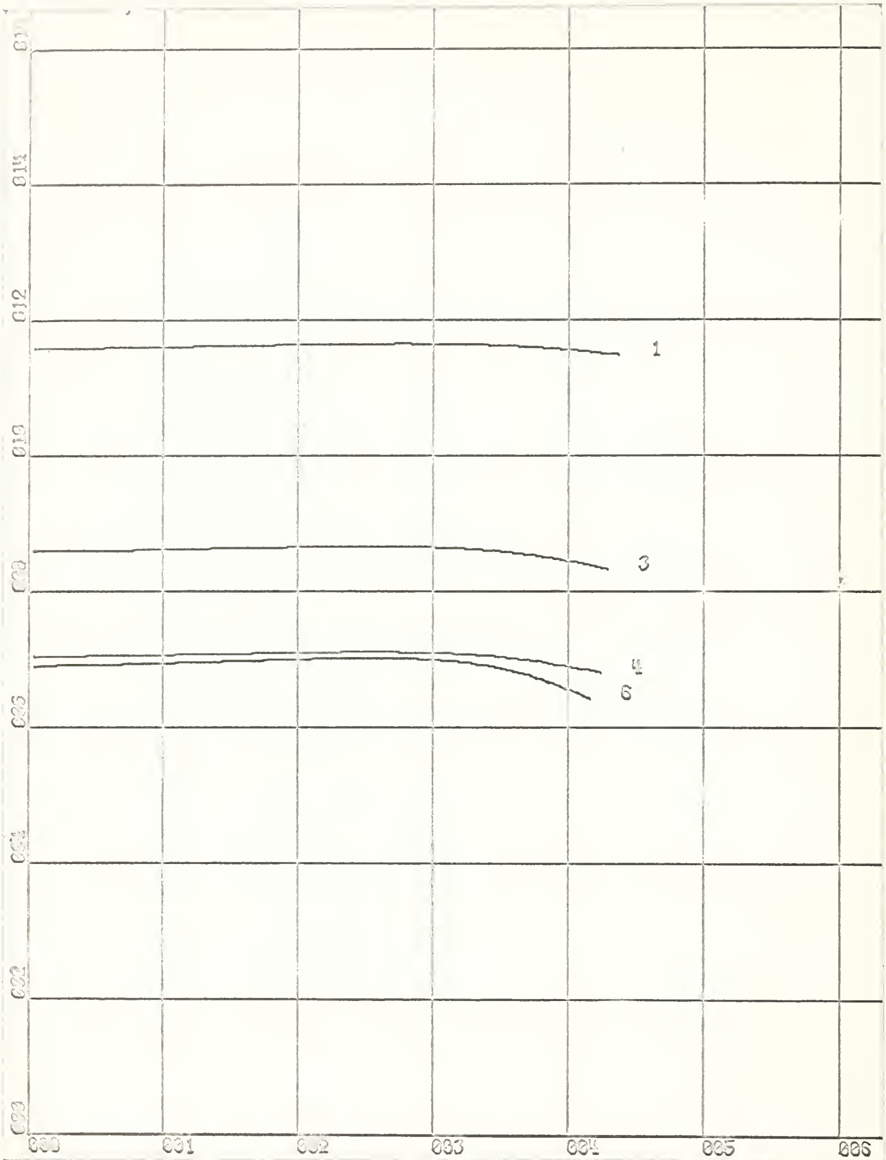
SUM OF VARIANCES VERSUS c



X-SCALE = 1.00E - 01 Units/Inch.
Y-SCALE = 5.00E - 02 Units/Inch.

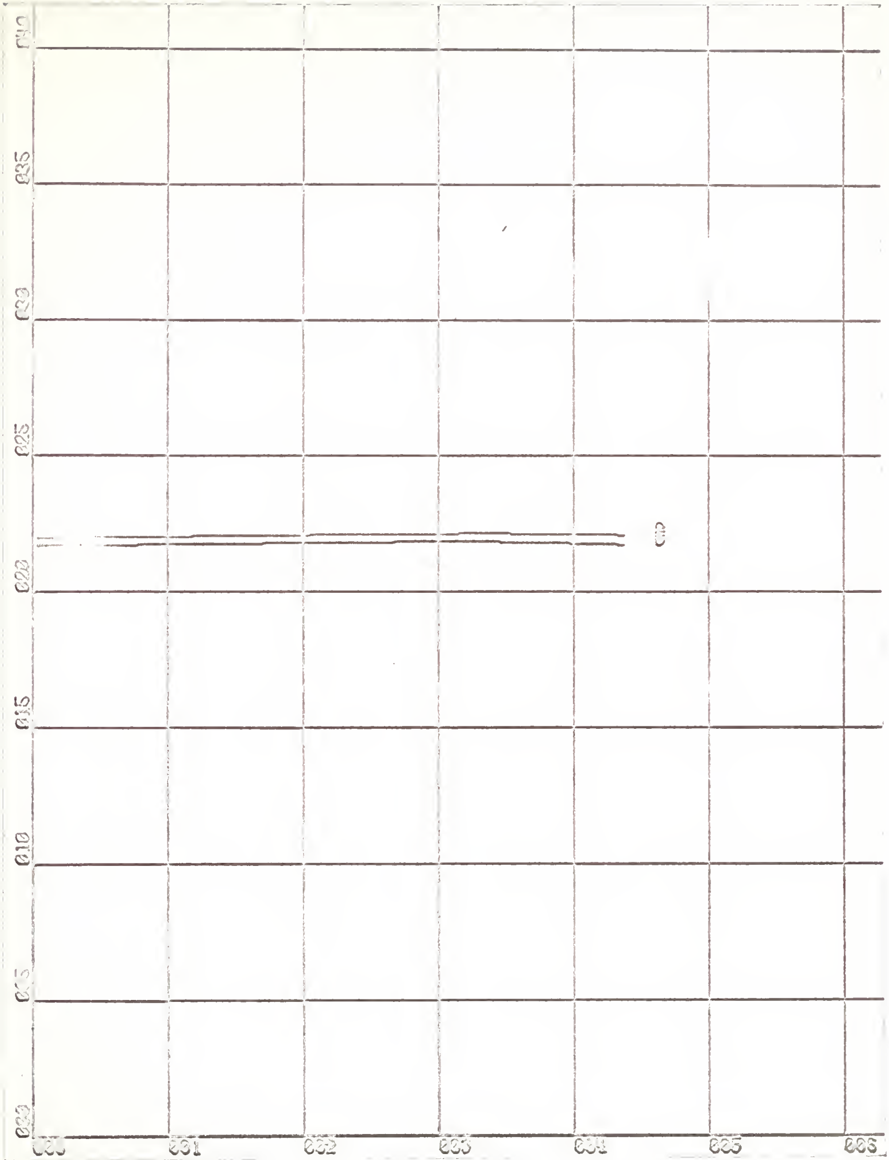
P(0) = 7.00E - 01
N = 3.00E + 02

SUM OF VARIANCES VERSUS c



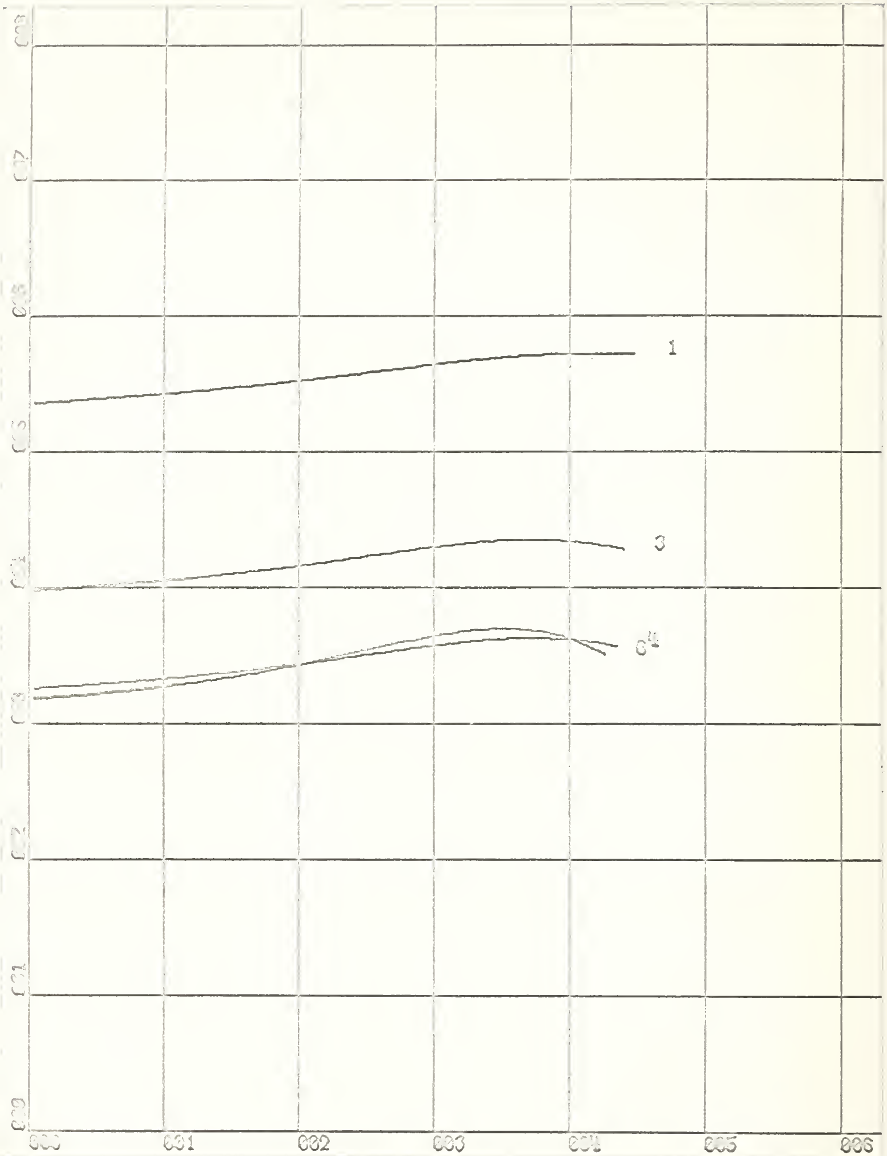
X-SCALE = 1.00E - 01 Units/Inch. P(0) = 8.00E - 01
Y-SCALE = 2.00E - 02 Units/Inch. N = 3.00E + 02

SUM OF VARIANCES VERSUS c



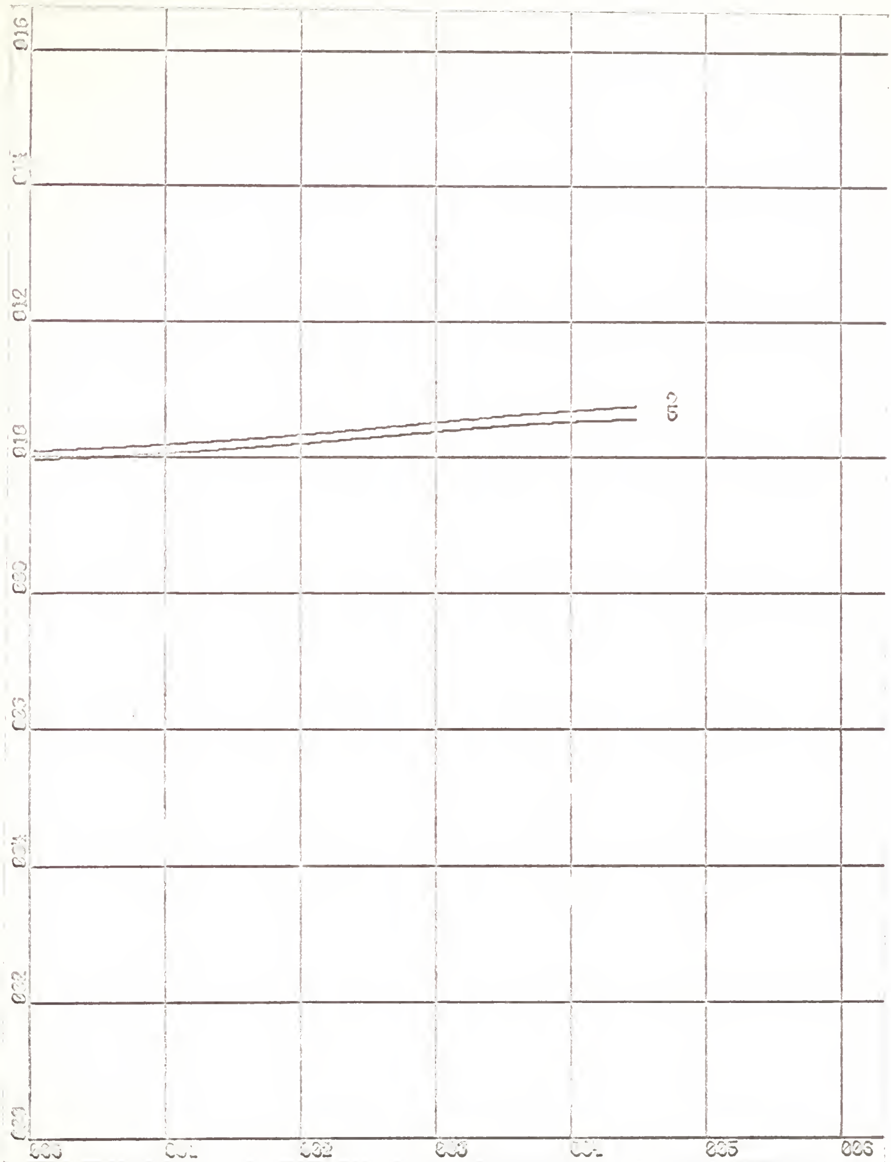
X-SCALE = 1.00E - 01 Units/Inch. P(0) = 8.00E - 01
Y-SCALE = 5.00E - 02 Units/Inch. N = 3.00E + 02

SUM OF VARIANCES VERSUS c



X-SCALE = 1.00E - 01 Units/Inch. P(0) = 9.00E - 01
Y-SCALE = 1.00E - 02 Units/Inch. N = 3.00E + 02

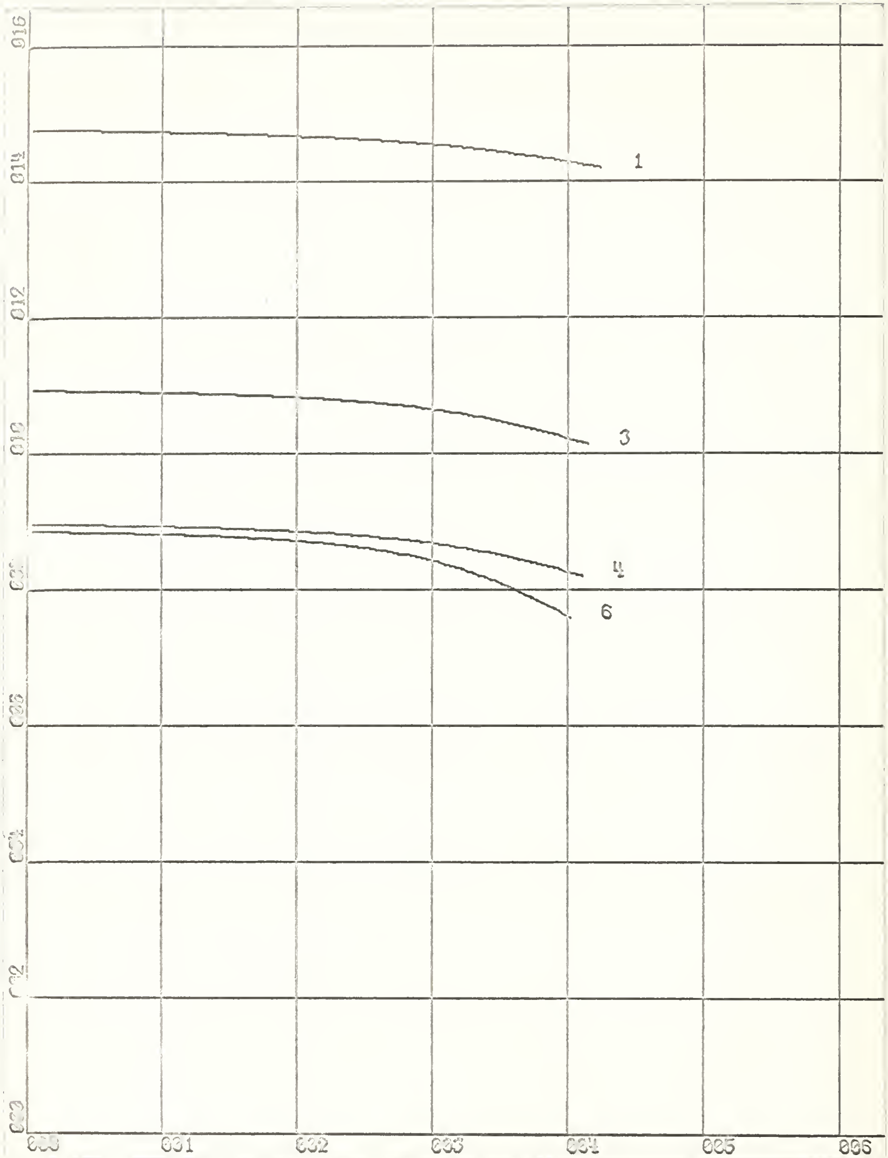
SUM OF VARIANCES VERSUS c



X-SCALE = 1.00E - 01 Units/Inch.
Y-SCALE = 2.00E - 02 Units/Inch.

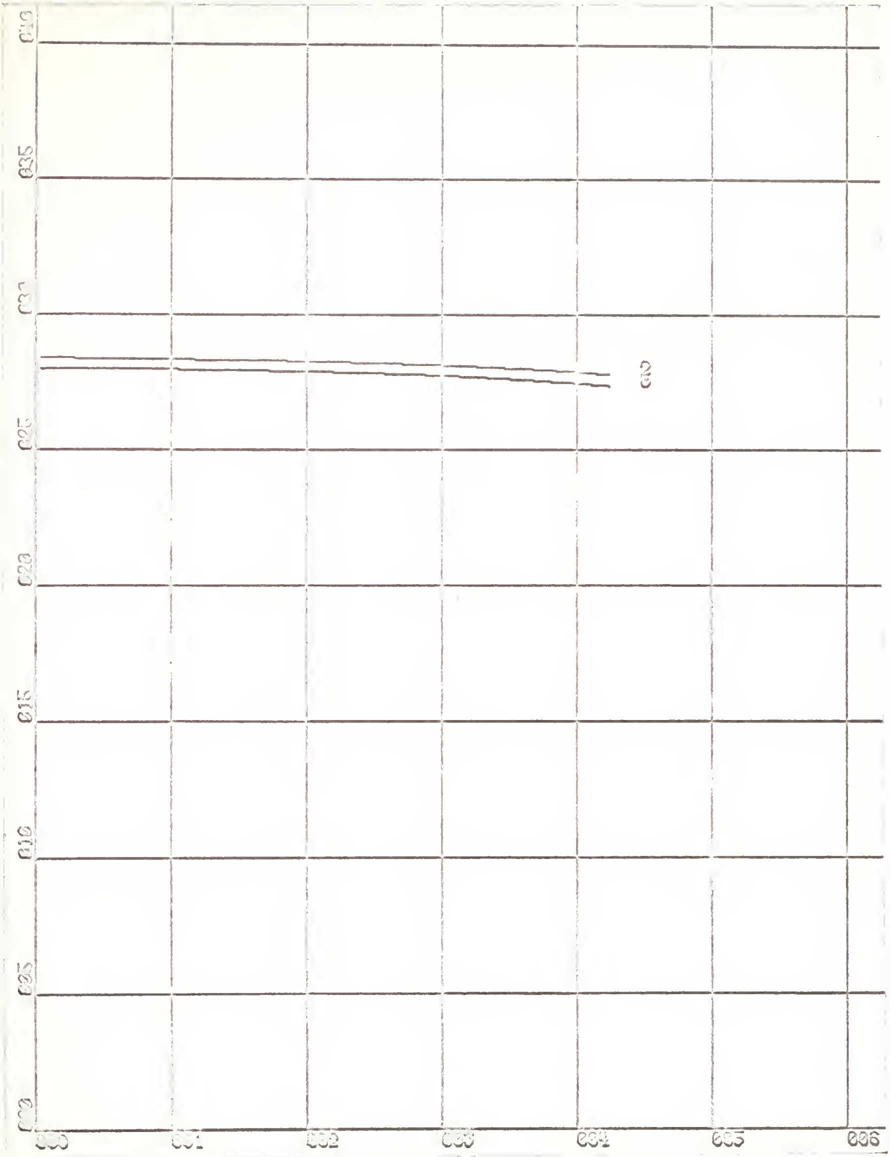
P(0) = 9.00E - 01
N = 3.00E + 02

SUM OF VARIANCES VERSUS c



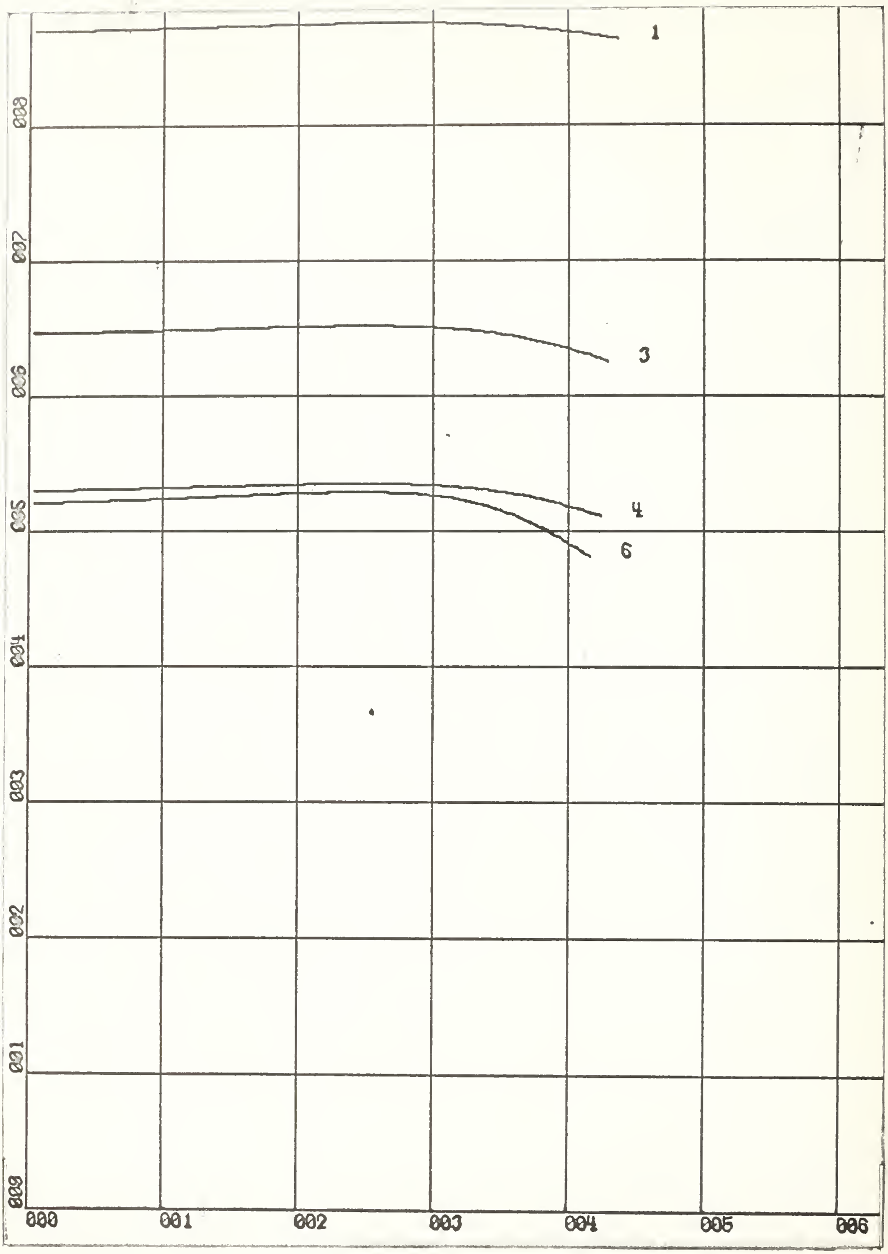
X-SCALE = 1.00E - 01 Units/Inch. P(0) = 7.00E - 01
Y-SCALE = 2.00E - 02 Units/Inch. N = 4.00E + 02

SUM OF VARIANCES VERSUS c



X-SCALE = 1.00E - 01 Units/Inch. P(0) = 7.00E - 01
Y-SCALE = 5.00E - 02 Units/Inch. N = 4.00E + 02

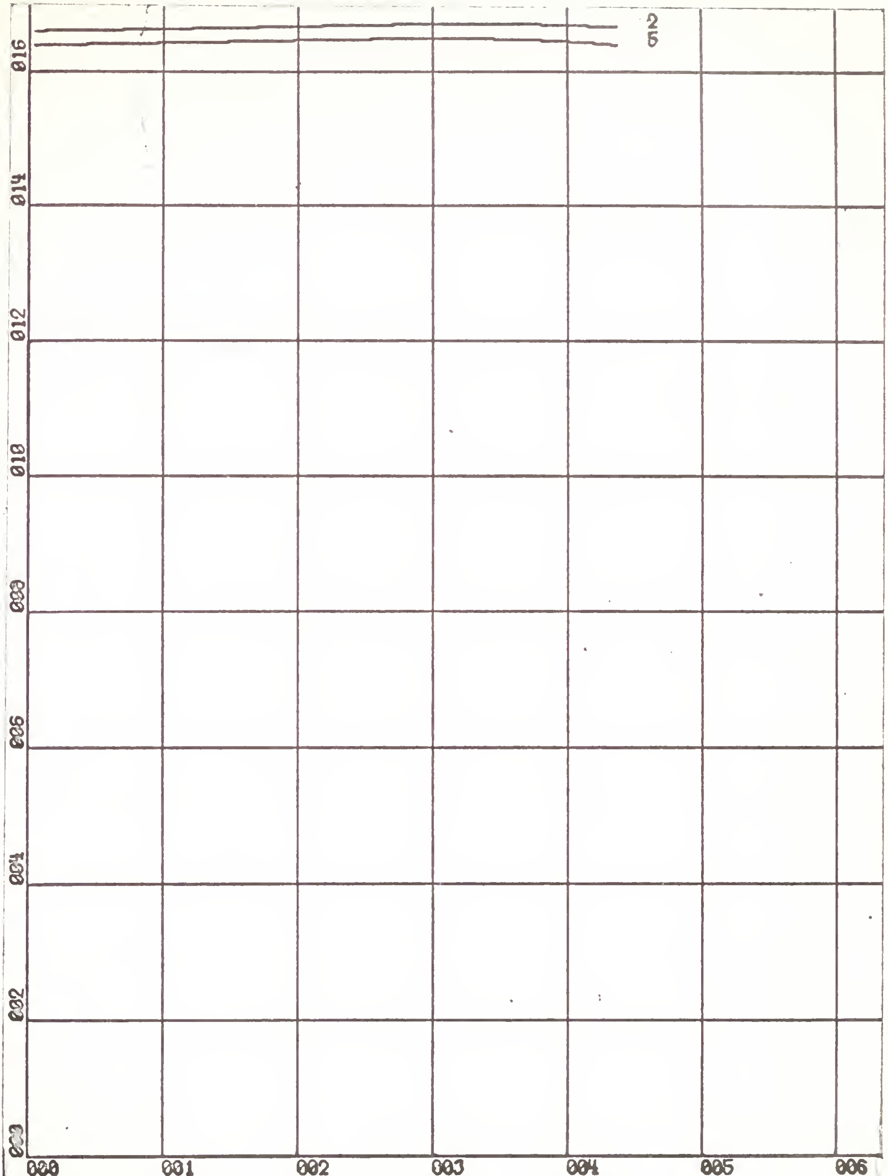
SUM OF VARIANCES VERSUS c



X-SCALE = 1.00E - 01 Units/Inch.
Y-SCALE = 1.00E - 02 Units/Inch.

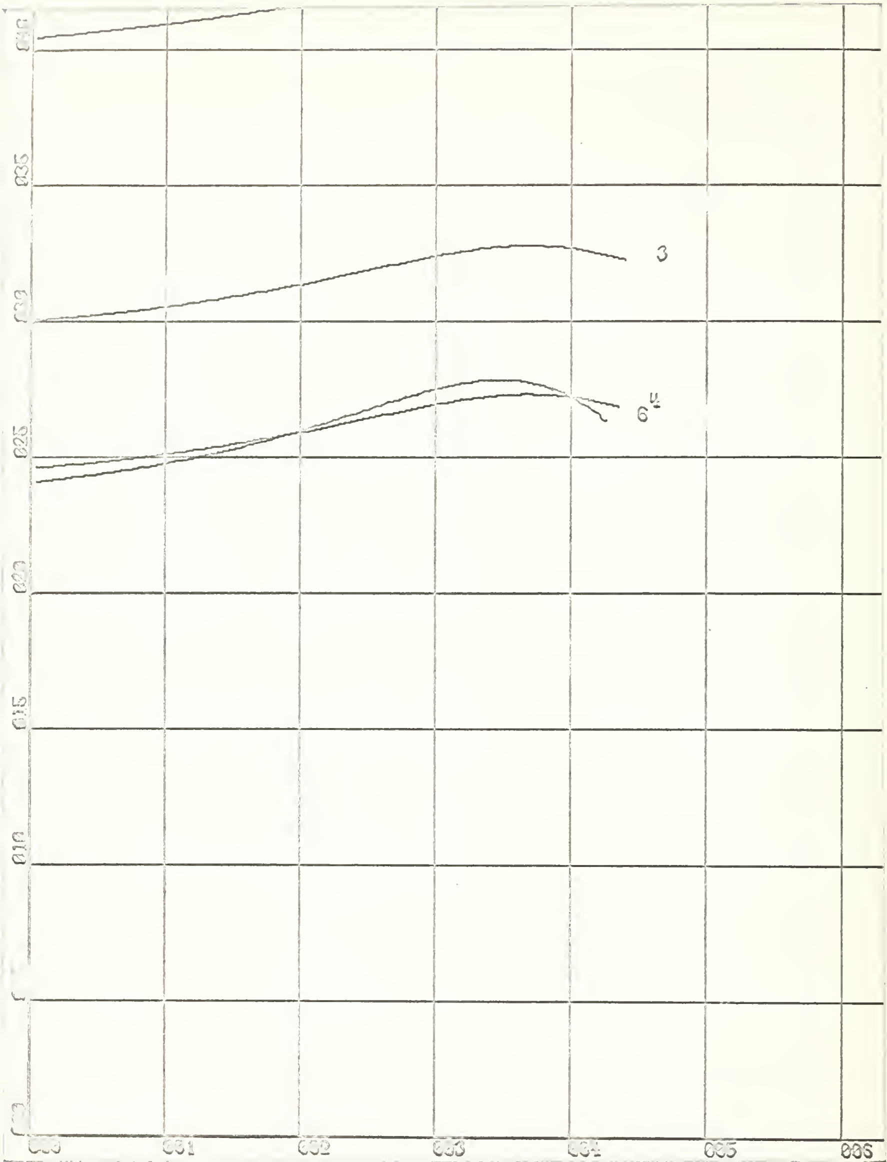
P(0) = 8.00E - 01
N = 4.00E + 02

SUM OF VARIANCES VERSUS c



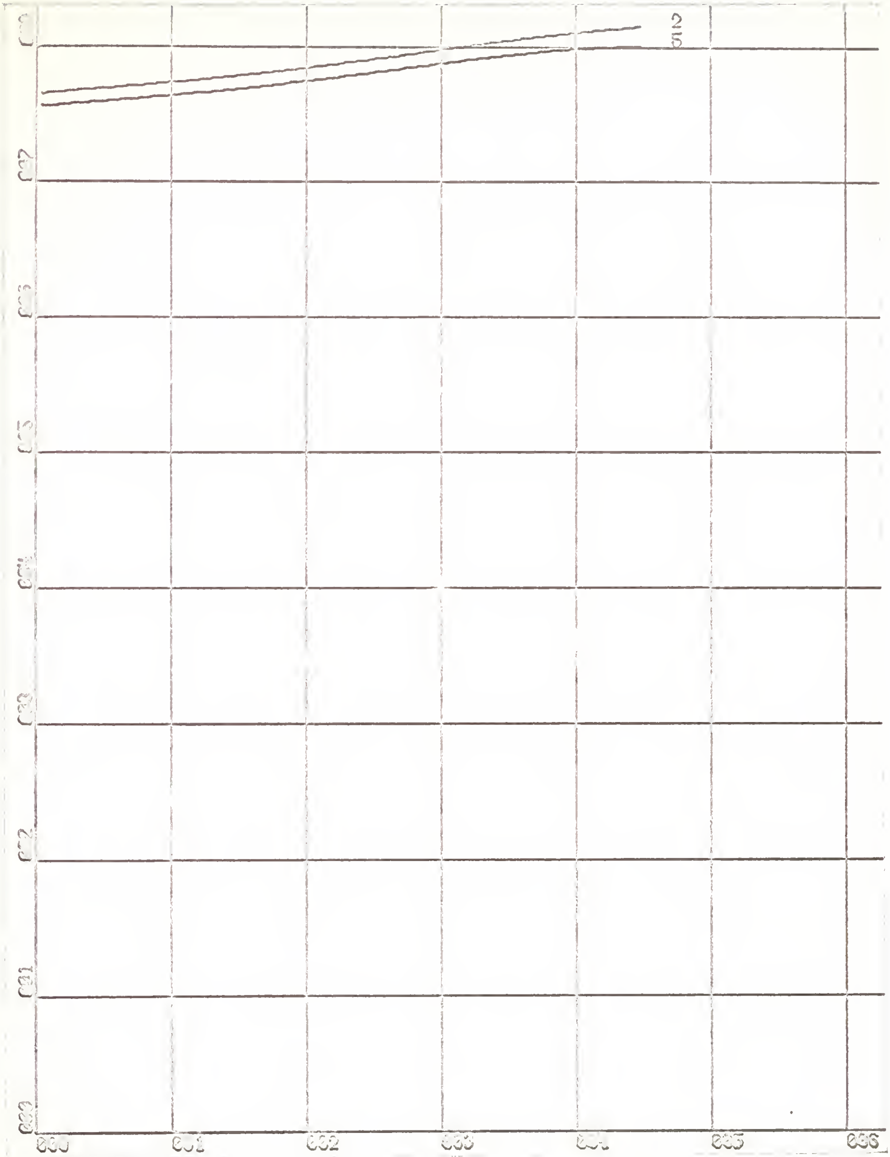
X-SCALE = 1.00E - 01 Units/Inch. P(0) = 8.00E - 01
Y-SCALE = 2.00E - 02 Units/Inch. N = 4.00E + 02

SUM OF VARIANCES VERSUS c



X-SCALE = 1.00E - 01 Units/Inch. P(0) = 9.00E - 01
Y-SCALE = 5.00E - 03 Units/Inch. N = 4.00E + 02

SUM OF VARIANCES VERSUS c



X-SCALE = 1.00E - 01 Units/Inch. P(0) = 9.00E - 01
Y-SCALE = 1.00E - 02 Units/Inch. N = 4.00E + 02

SUM OF VARIANCES VERSUS c

2.5 Determining FOT Sample Size Using Model Three

Model Three uses the following model for reliability at any time t:

$$R(t) = p_0 - at$$

where:

p_0 = the initial reliability;

a = an unknown parameter.

This linear model was chosen so that the more complex models one and two might be compared with a simple and familiar model.

To investigate this model, a maximum likelihood estimator for reliability was obtained. Next, an expression for the variance of this estimator was determined, see section 3.5. Using the method outlined in section 2.2, the sum of the variances was then computed as a function of a for all combinations of p_0 and N. The values of the sum of the variances were then determined for a range of values of a between 0 and $p_0/10$. At this point, R(10) becomes negative; therefore, the sum of the variances has been computed for the entire range of plausible values of a.

The results of the computations of the sum of the variances are displayed on the graphs on pages 63 through 80. The abscissa of the graph represents values of a and the ordinate represents values of the sum of the variances. See page 17 for an explanation of how to read these graphs.

By examining these graphs, it can be seen that the sample plan that gives the minimum sum of the variances for all combinations of p_0, N, and values of a is sample plan six. The value of the sum of the variances increases for each sample plan in the following order: plan four, plan three, plan one, plan five, and then plan two.

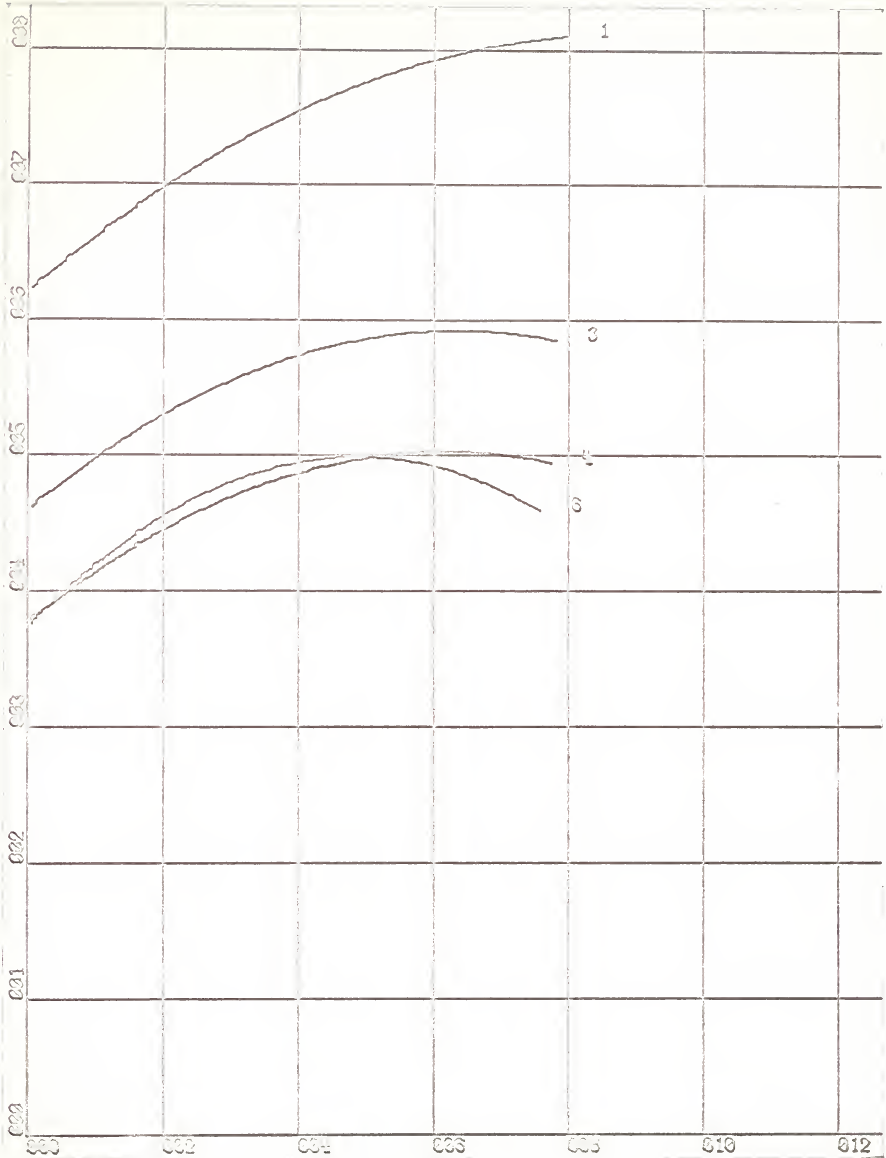
For convenience, a table explaining the size of each yearly test under the different sample plans is presented on the following page. Notice that as in the first two models the minimum sum of the variances occurs for the plans that test heavily in the early years and moderately in the out years. The plans that call for heavy testing in the out years have the largest values of the sum of the variances.

Sample Plan No.	Year 1	Year 2	Year 3	Year 4	Year 5	Year 6	Year 7	Year 8	Year 9	Year 10
1	.1N	.1N	.1N	.1N	.1N	.1N	.1N	.1N	.1N	.1N
2	.05N	.05N	.05N	.05N	.05N	.15N	.15N	.15N	.15N	.15N
3	.15N	.15N	.15N	.15N	.15N	.05N	.05N	.05N	.05N	.05N
4	.3N	.1N	.1N	.1N	.1N	.1N	.05N	.05N	.05N	.05N
5	.05N	.05N	.05N	.05N	.3N	.1N	.1N	.1N	.1N	.1N
6	.3N	.2N	.1N	.1N	.1N	.1N	.03N	.03N	.02N	.02N

(N = the total number of missiles tested in the first ten years of operation.)

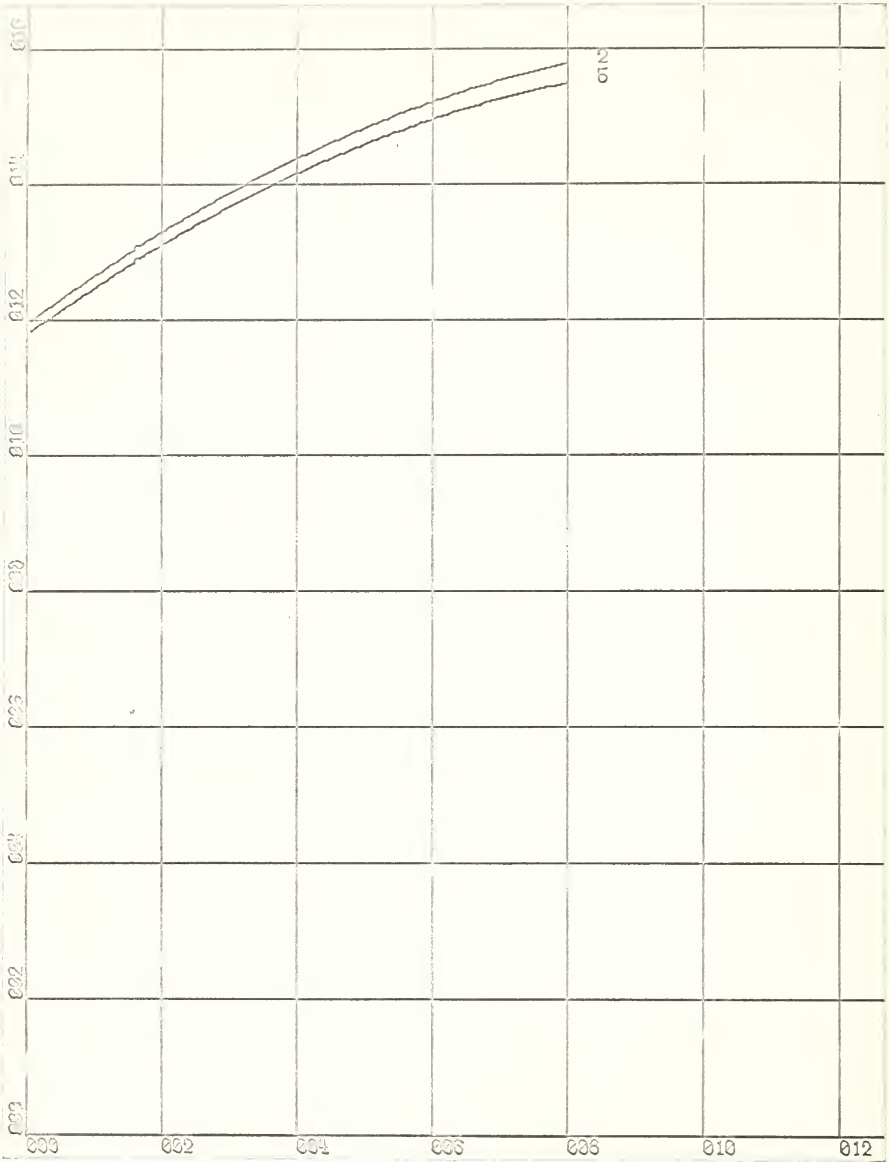
YEARLY TRIALS IN SAMPLE PLANS ONE THROUGH SIX

FIGURE 2. 5. 1



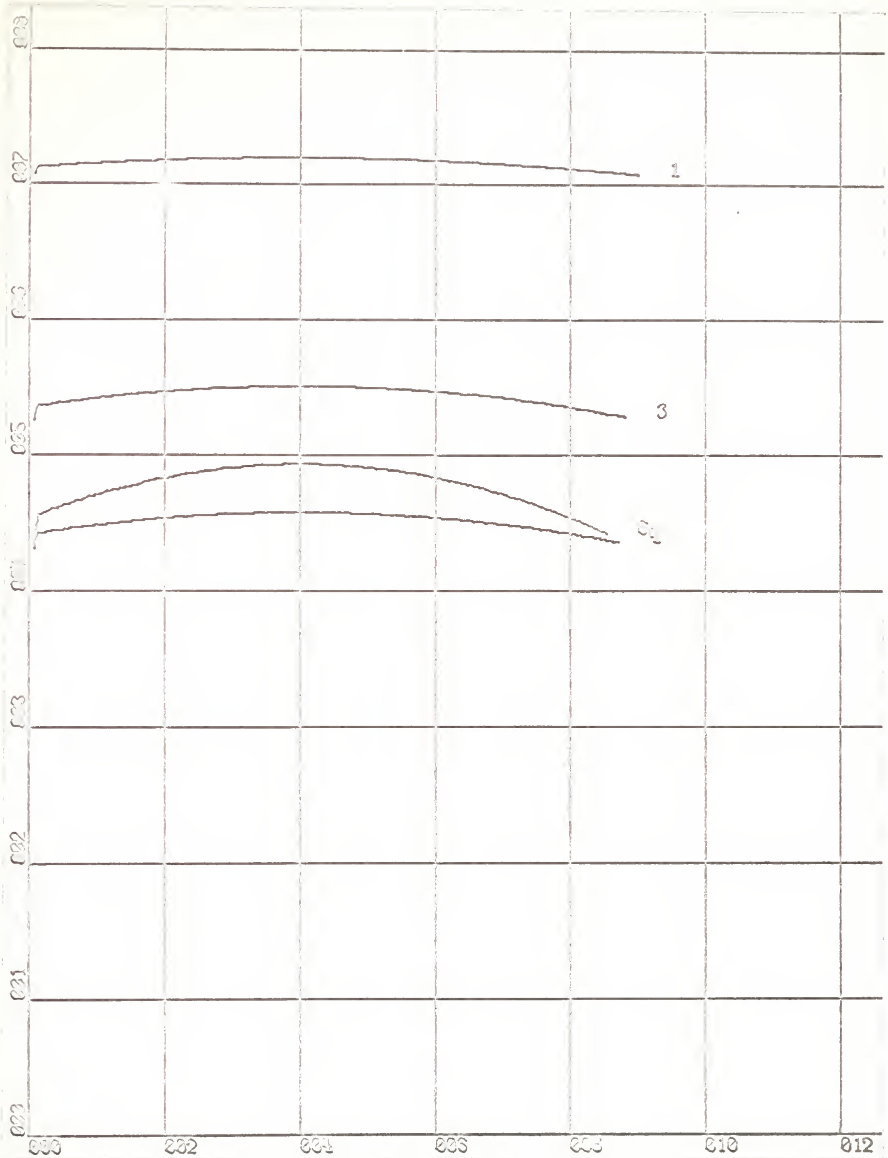
X-SCALE = 2.00E - 02 Units/Inch. P(0) = 8.00E - 01
Y-SCALE = 1.00E - 03 Units/Inch. N = 4.00E + 02

SUM OF VARIANCES VERSUS a



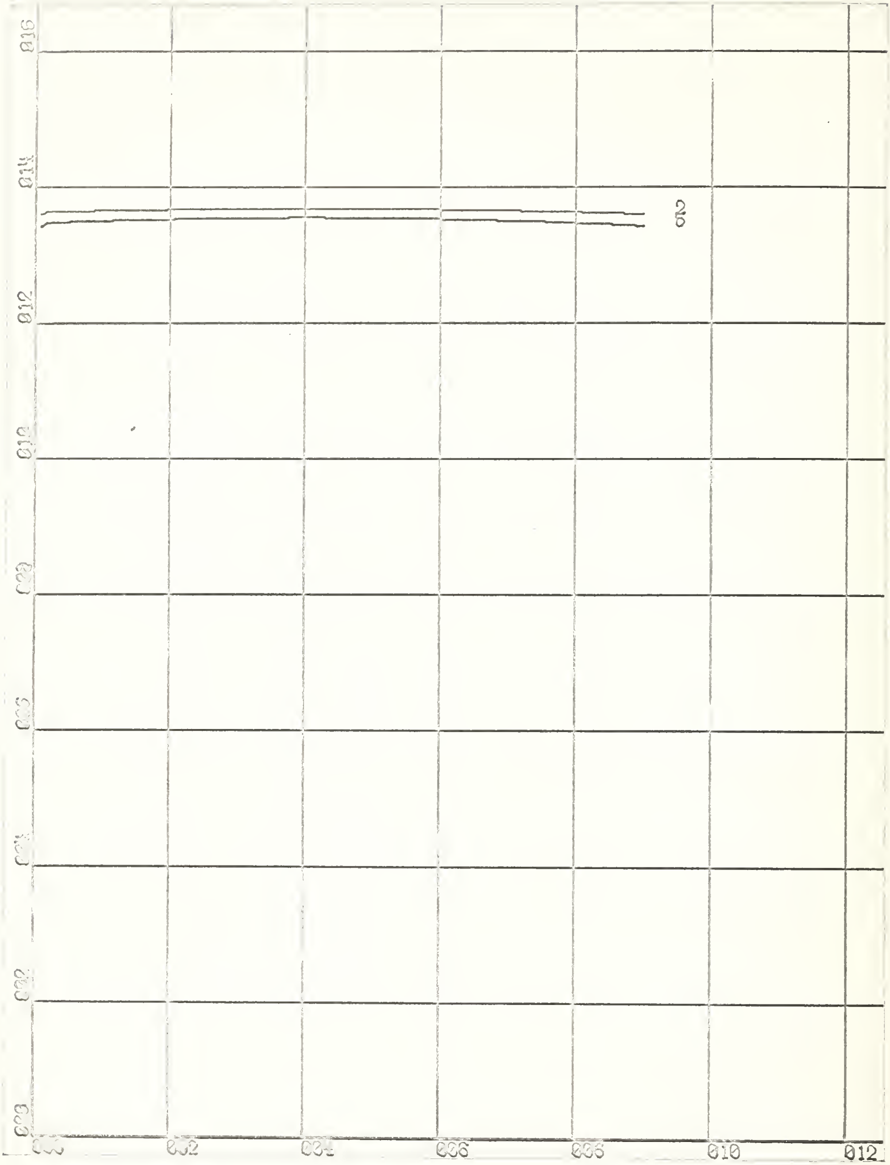
X-SCALE = 2.00E - 02 Units/Inch. P(0) = 8.00E - 01
Y-SCALE = 2.00E - 03 Units/Inch. N = 4.00E + 02

SUM OF VARIANCES VERSUS a



X-SCALE = 2.00E - 02 Units/Inch. P(0) = 9.00E - 01
Y-SCALE = 1.00E - 03 Units/Inch. N = 2.00E + 02

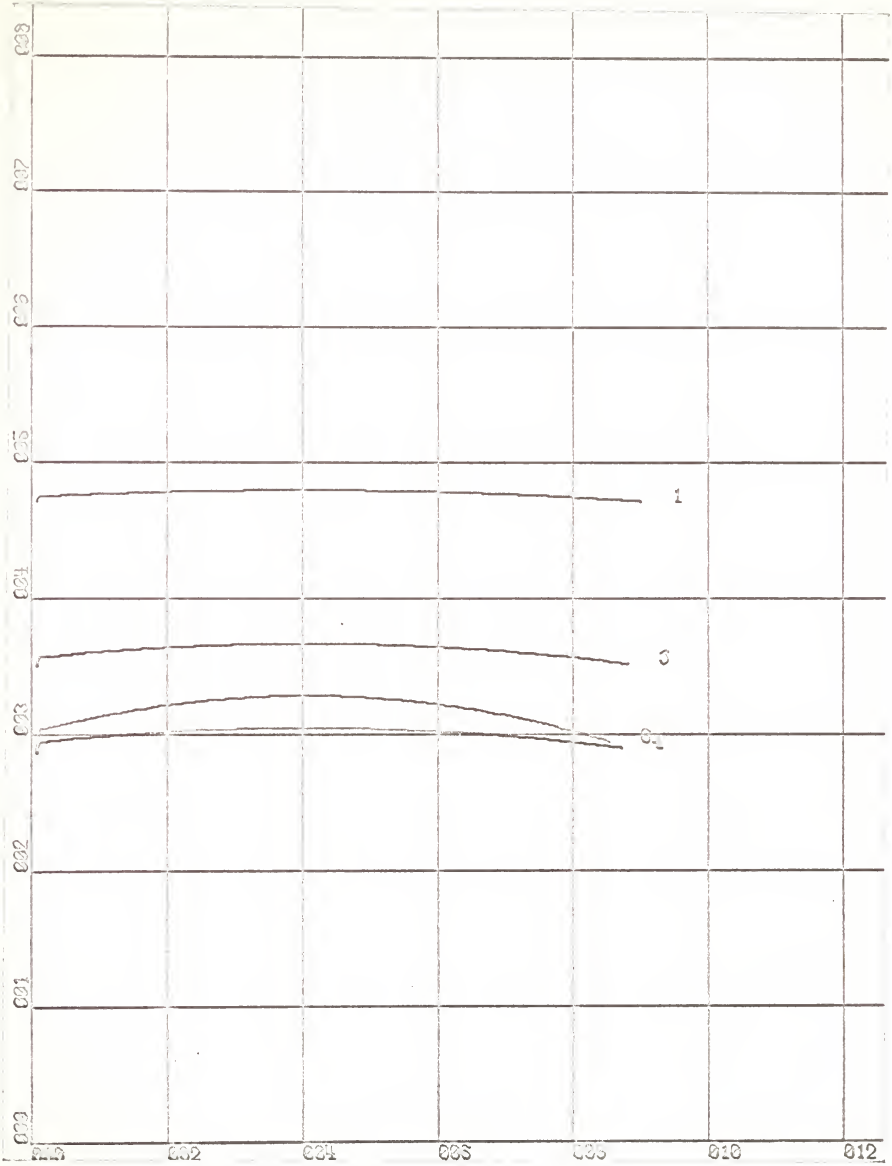
SUM OF VARIANCES VERSUS a



X-SCALE = 2.00E - 02 Units/Inch.
Y-SCALE = 2.00E - 03 Units/Inch.

P(0) = 9.00E - 01
N = 2.00E + 02

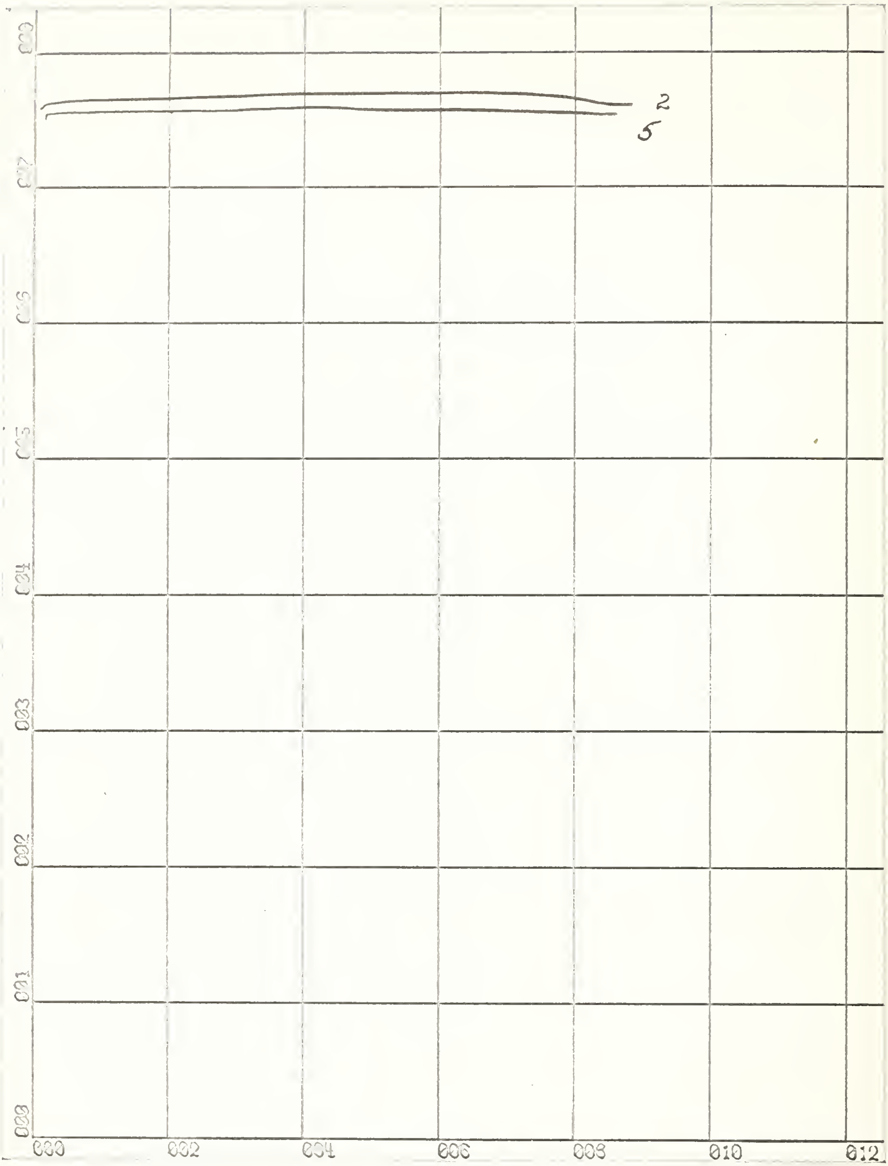
SUM OF VARIANCES VERSUS a



X-SCALE = 2.00E - 02 Units/Inch.
Y-SCALE = 1.00E - 03 Units/Inch.

P(0) = 9.00E - 01
N = 3.00E + 02

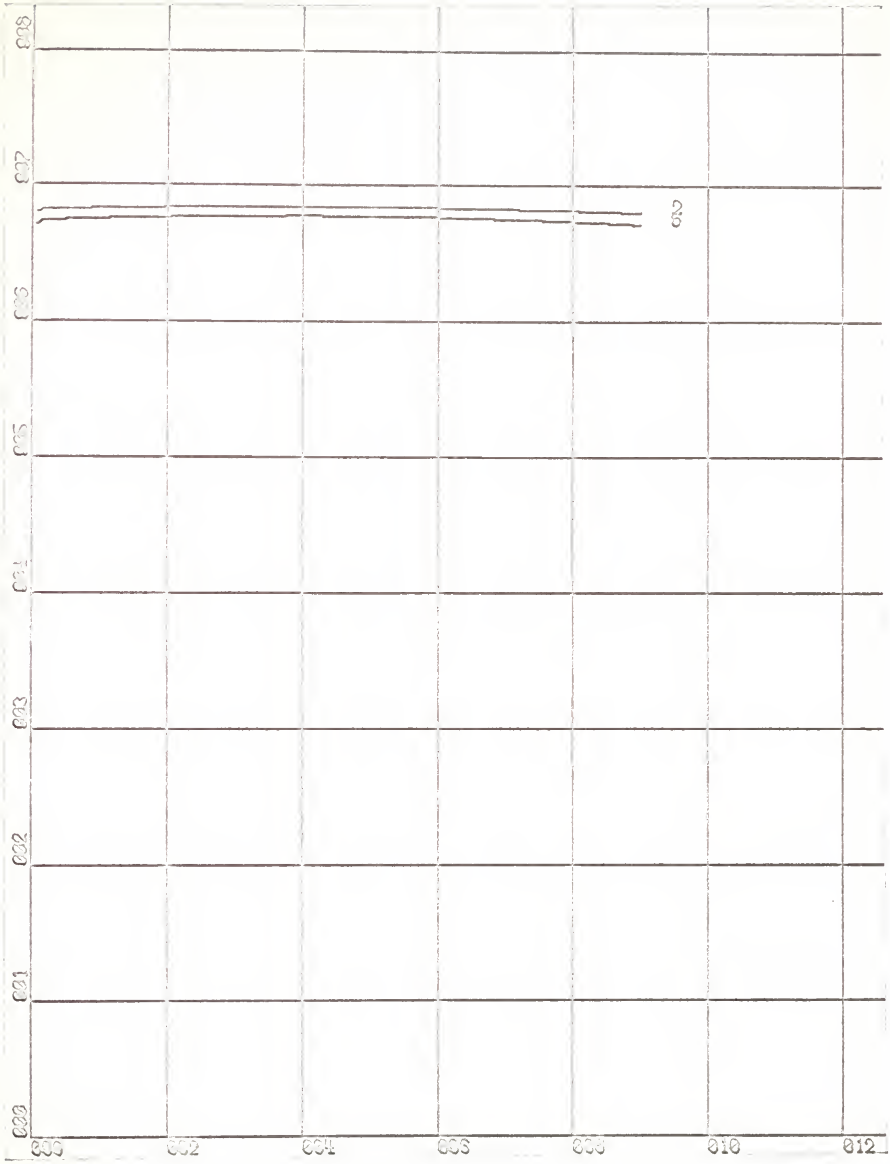
SUM OF VARIANCES VERSUS a



X-SCALE = 2.00E - 02 Units/Inch.
Y-SCALE = 1.00E - 03 Units/Inch.

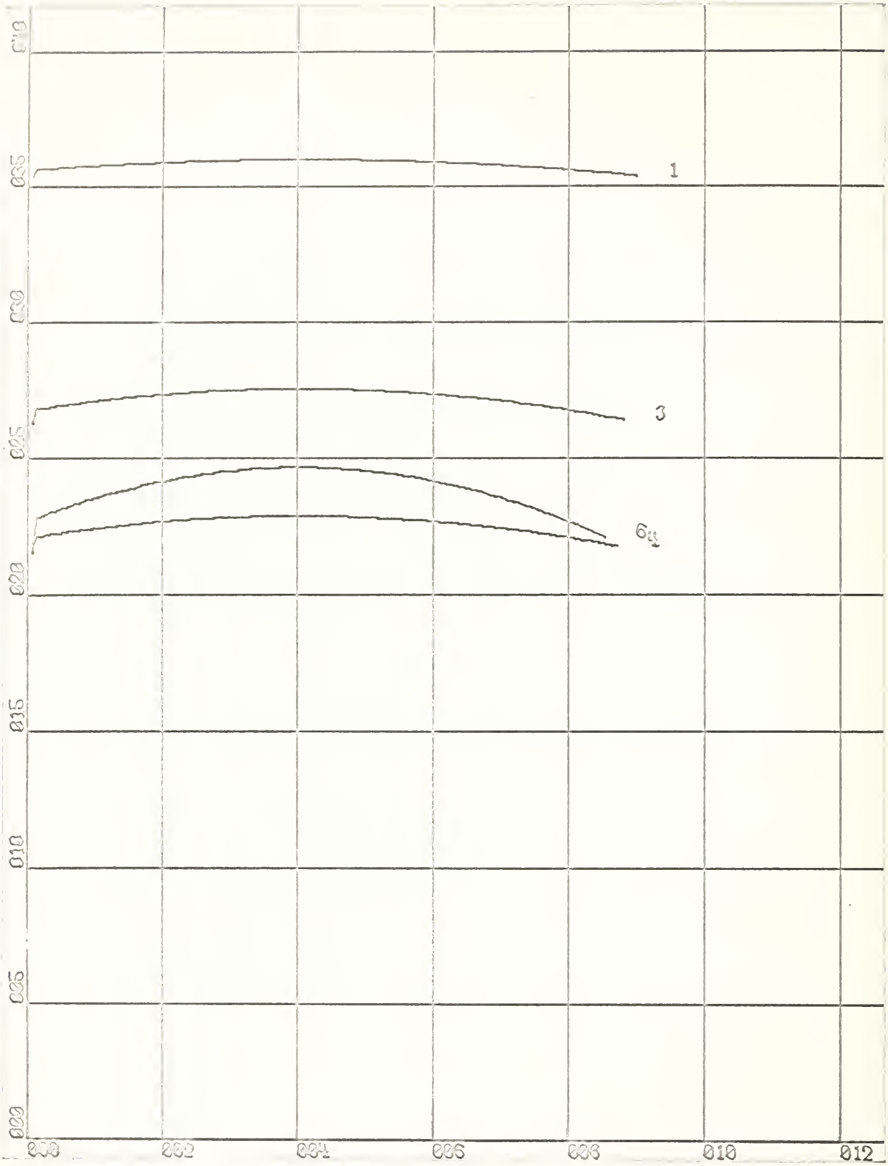
P(0) = 9.00E - 01
N = 3.00E + 02

SUM OF VARIANCES VERSUS a



X-SCALE = 2.00E - 02 Units/Inch. P(0) = 9.00E - 01
Y-SCALE = 1.00E - 03 Units/Inch. N = 4.00E + 02

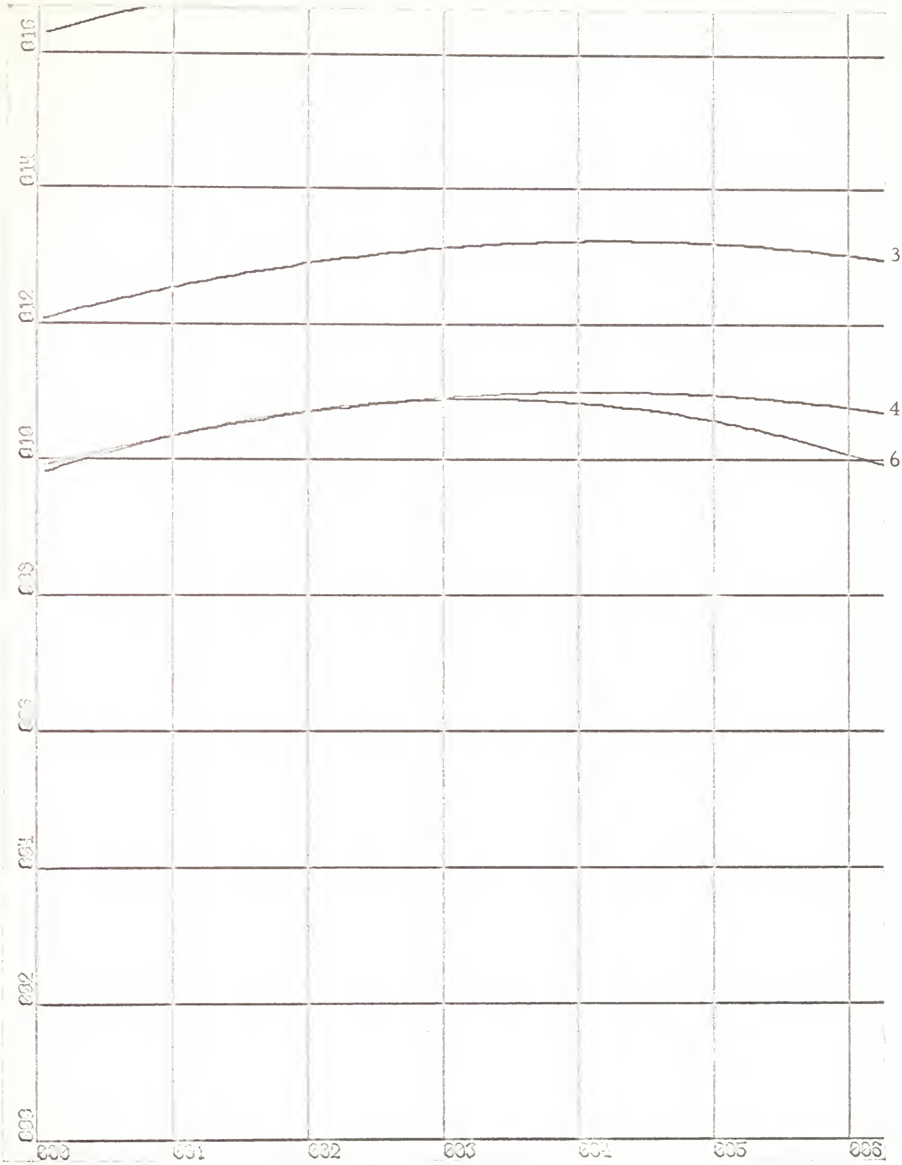
SUM OF VARIANCES VERSUS a



X-SCALE = 2.00E - 02 Units/Inch.
Y-SCALE = 5.00E - 04 Units/Inch.

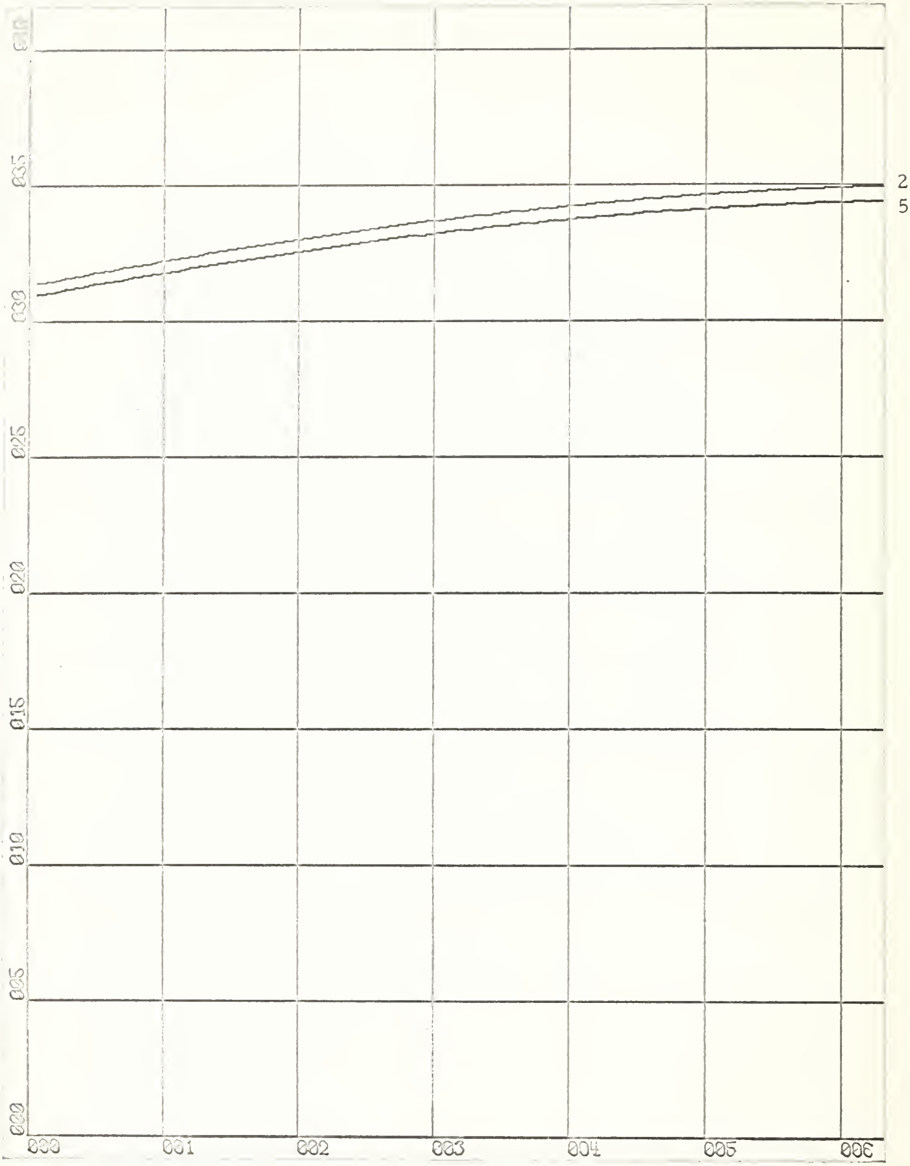
P(0) = 9.00E - 01
N = 4.00E + 02

SUM OF VARIANCES VERSUS a



X-SCALE = 1.00E - 02 Units/Inch. P(0) = 7.00E - 01
Y-SCALE = 2.00E - 03 Units/Inch. N = 2.00E + 02

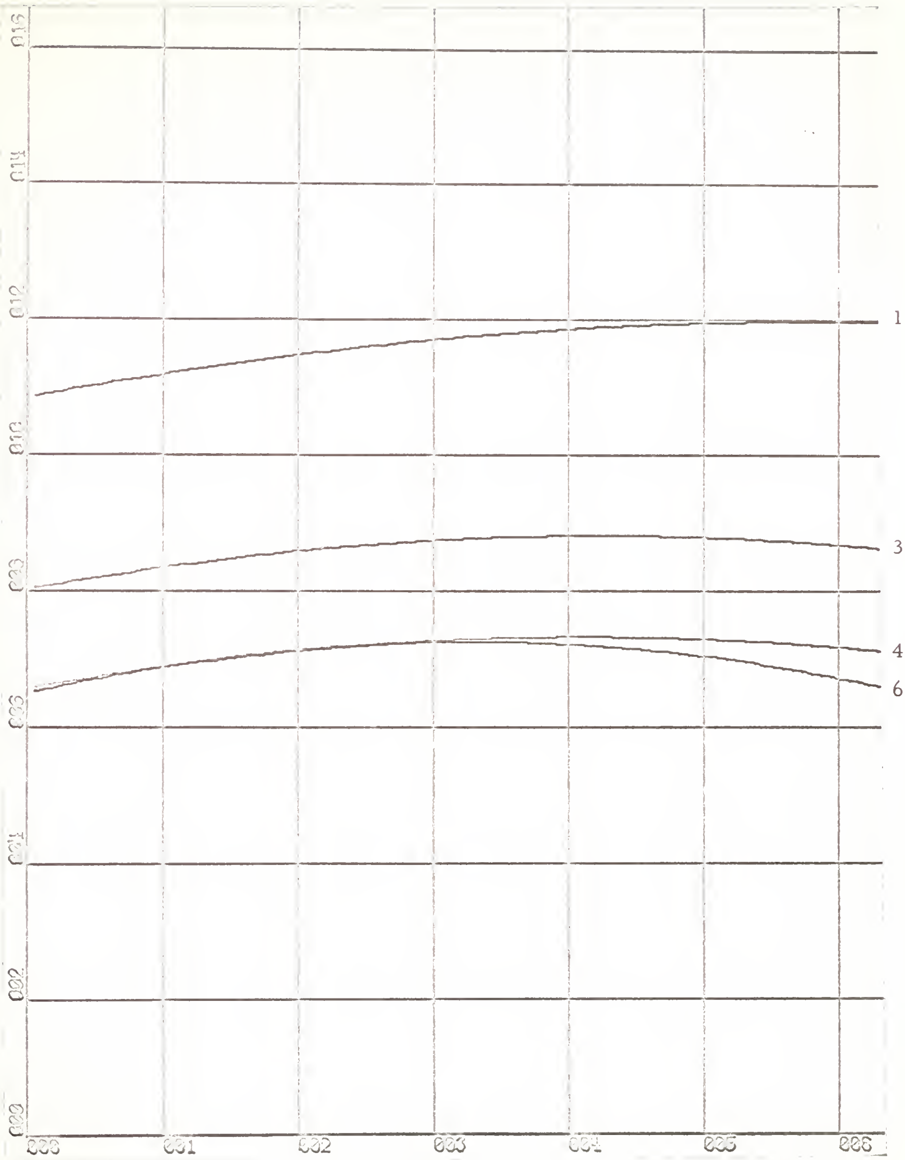
SUM OF VARIANCES VERSUS a



X-SCALE = 1.00E - 02 Units/Inch.
Y-SCALE = 5.00E - 03 Units/Inch.

P(0) = 7.00E - 01
N = 2.00E + 02

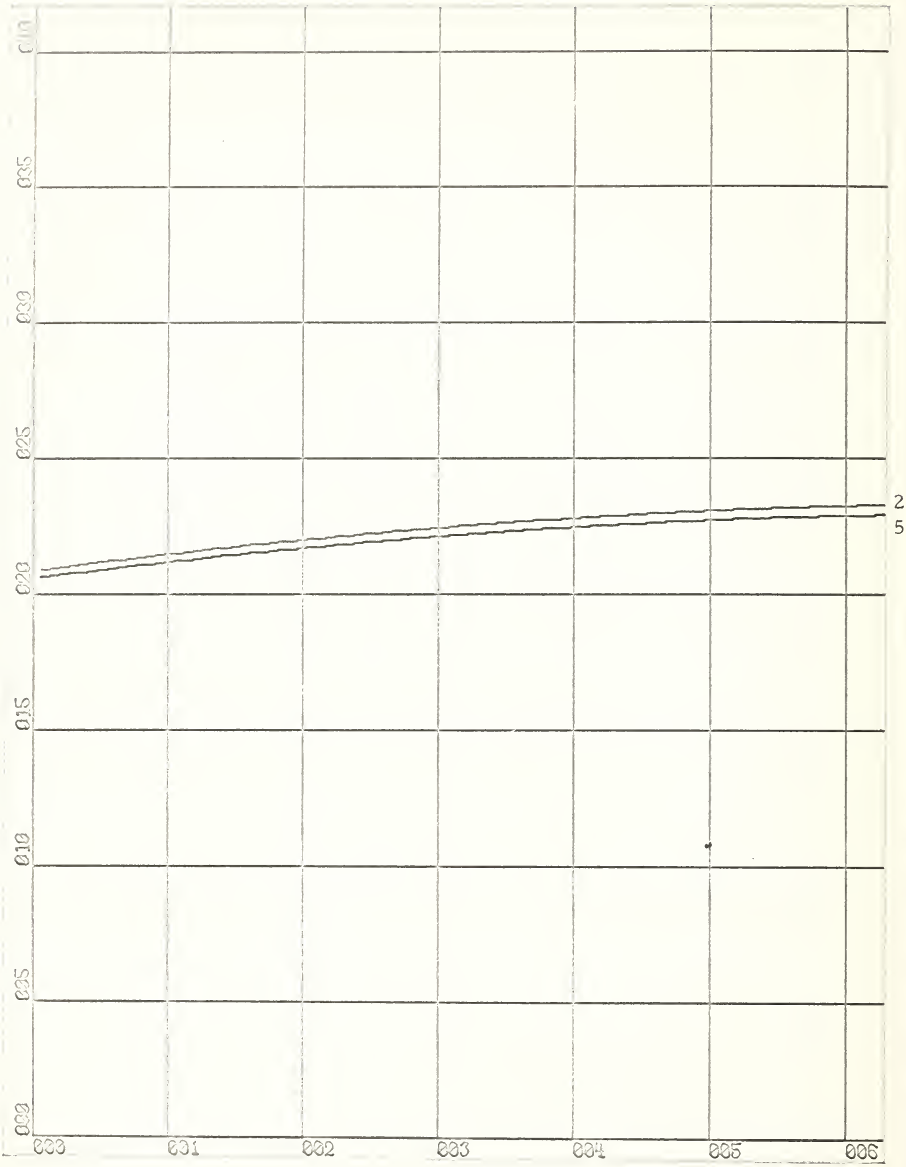
SUM OF VARIANCES VERSUS a



X-SCALE = 1.00E - 02 Units/Inch.
Y-SCALE = 2.00E - 03 Units/Inch.

P(0) = 7.00E - 01
N = 3.00E + 02

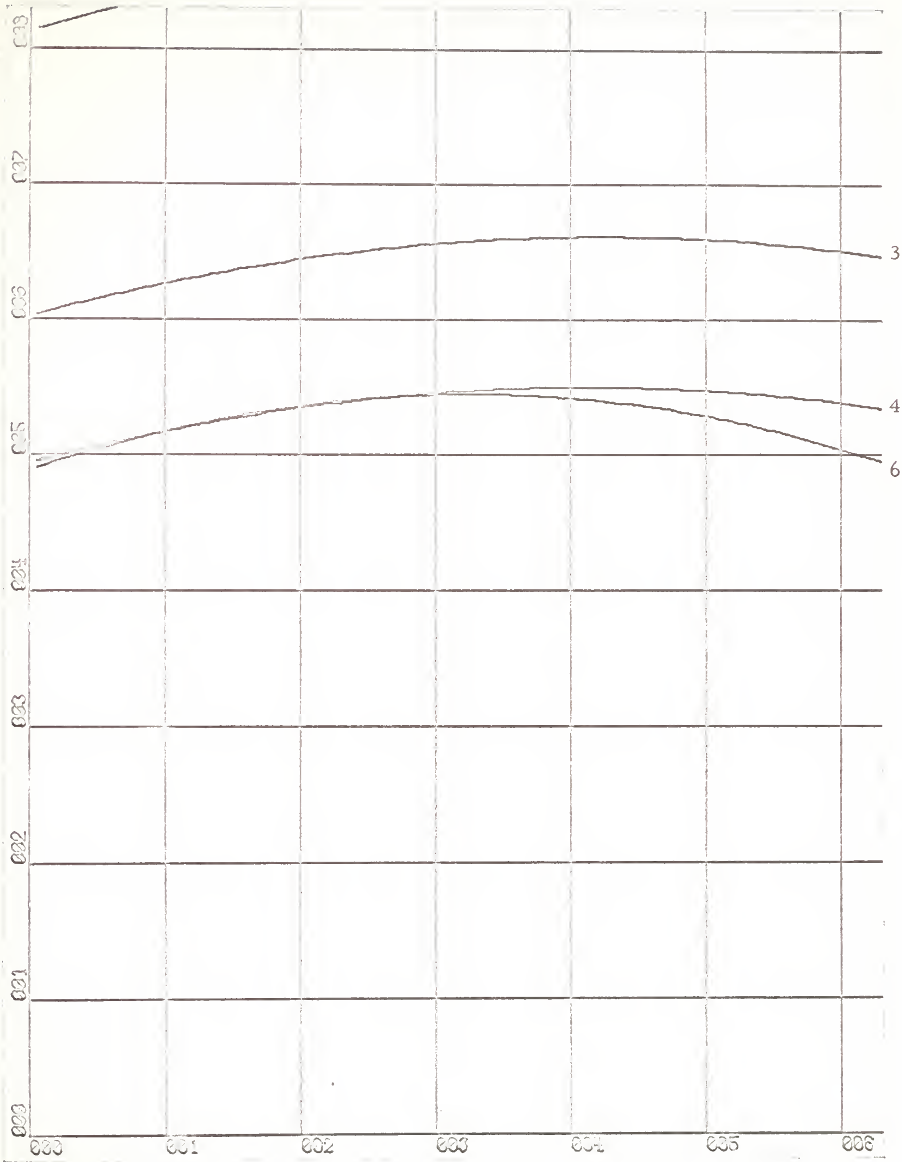
SUM OF VARIANCES VERSUS a



X-SCALE = 1.00E - 02 Units/Inch.
Y-SCALE = 5.00E - 03 Units/Inch.

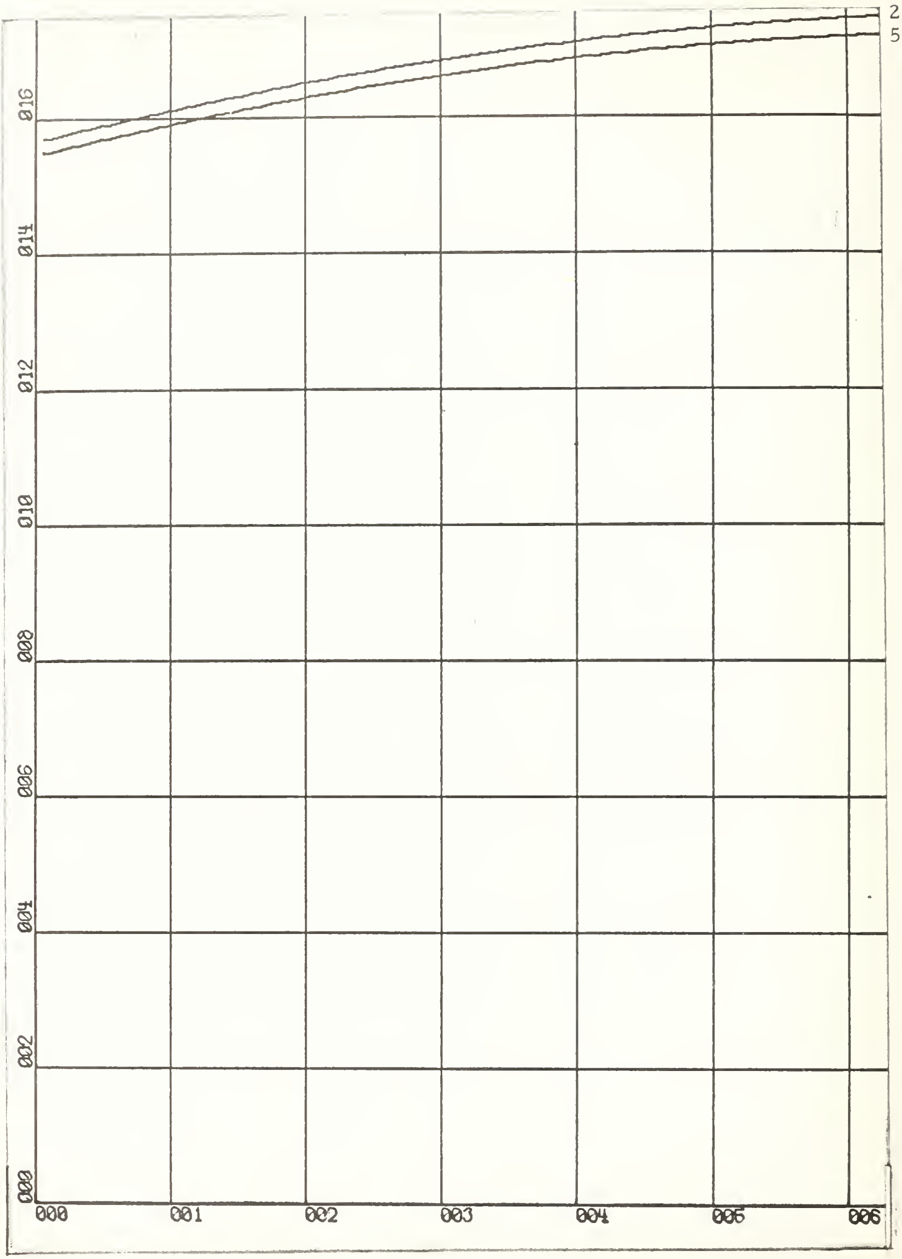
P(0) = 7.00E - 01
N = 3.00E + 02

SUM OF VARIANCES VERSUS a



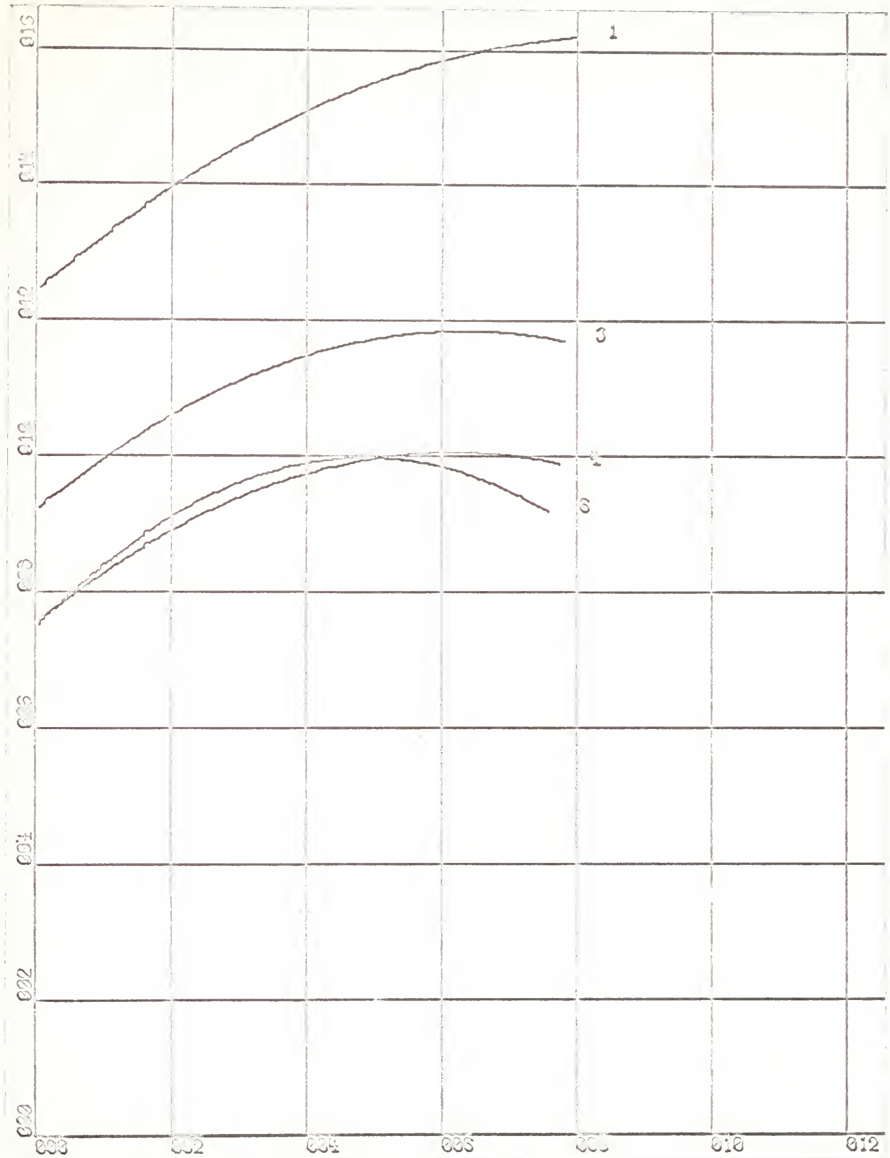
X-SCALE = 1.00E - 02 Units/Inch. P(0) = 7.00E - 01
Y-SCALE = 1.00E - 03 Units/Inch. N = 4.00E + 02

SUM OF VARIANCES VERSUS a



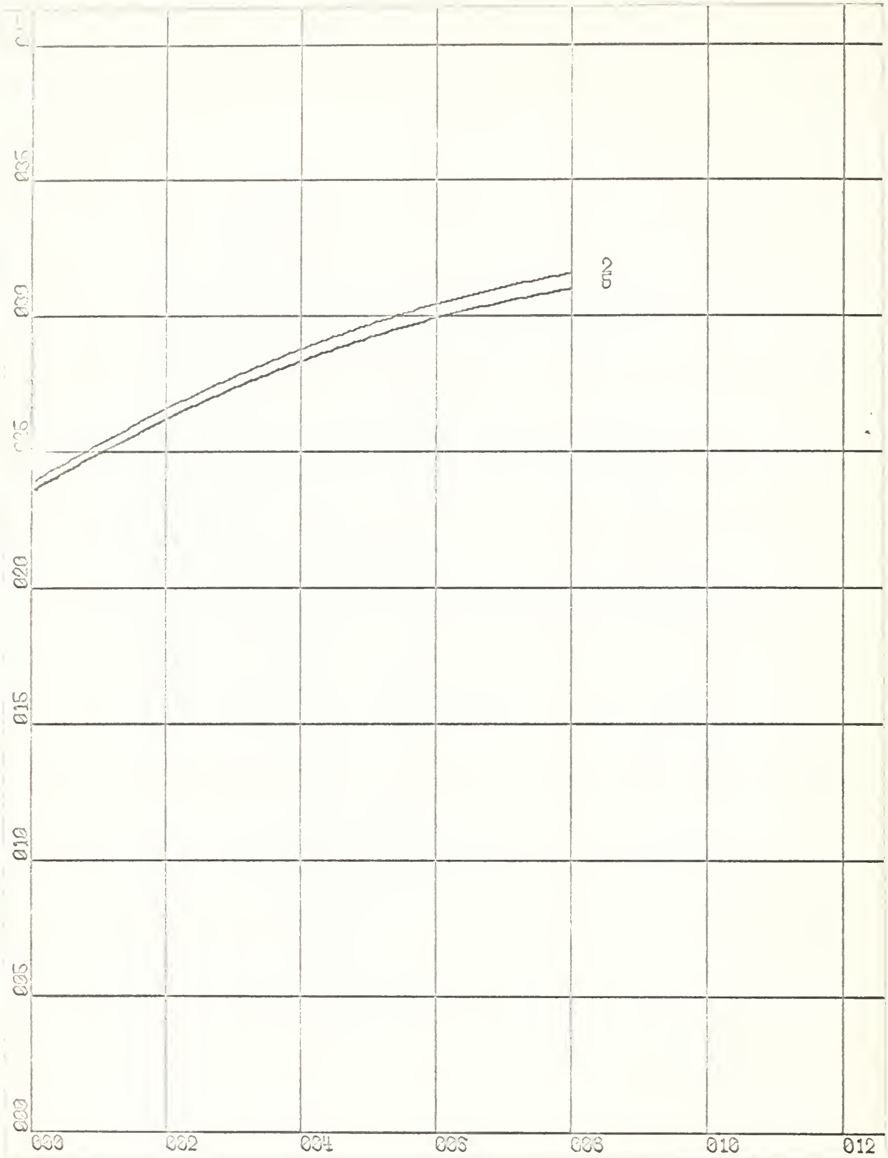
X-SCALE = 1.00E - 02 Units/Inch. P(0) = 7.00E - 01
Y-SCALE = 2.00E - 03 Units/Inch. N = 4.00E + 02

SUM OF VARIANCES VERSUS a



X-SCALE = 2.00E - 02 Units/Inch. P(0) = 8.00E - 01
Y-SCALE = 2.00E - 03 Units/Inch. N = 2.00E + 02

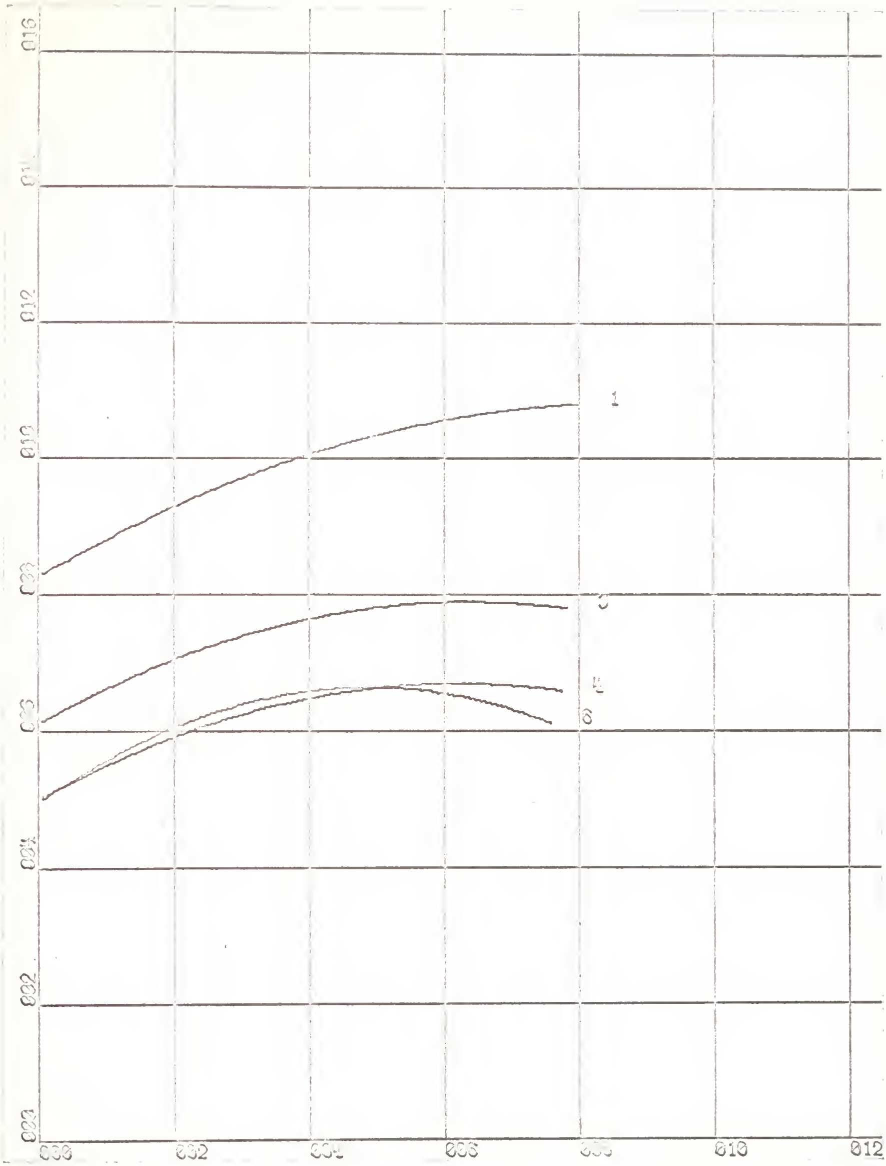
SUM OF VARIANCES VERSUS a



X-SCALE = 2.00E - 02 Units/Inch.
Y-SCALE = 5.00E - 03 Units/Inch.

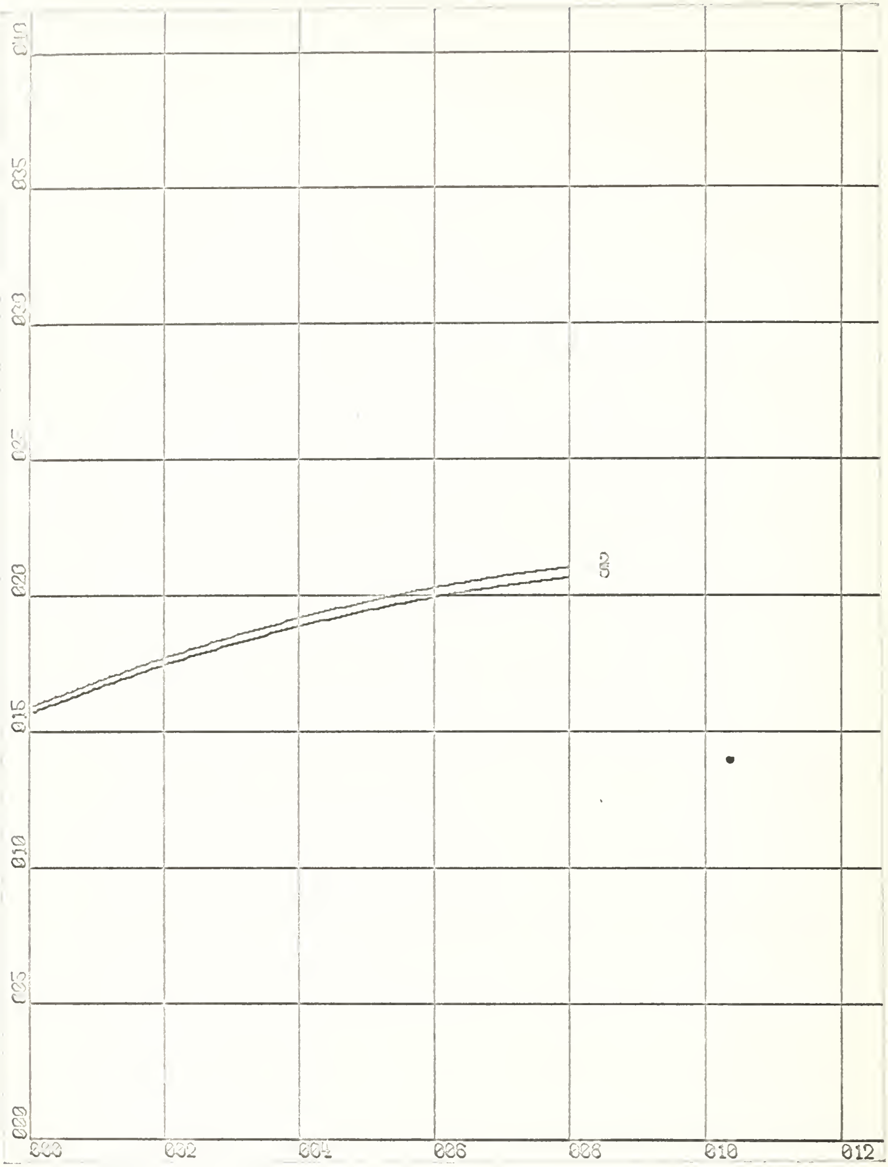
P(0) = 8.00E - 01
N = 2.00E + 02

SUM OF VARIANCES VERSUS a



X-SCALE = 2.00E - 02 Units/Inch. P(0) = 8.00E - 01
Y-SCALE = 2.00E - 03 Units/Inch. N = 3.00E + 02

SUM OF VARIANCES VERSUS a



X-SCALE = 2.00E - 02 Units/Inch.
Y-SCALE = 5.00E - 03 Units/Inch.

P(0) = 8.00E - 01
N = 3.00E + 02

SUM OF VARIANCES VERSUS a

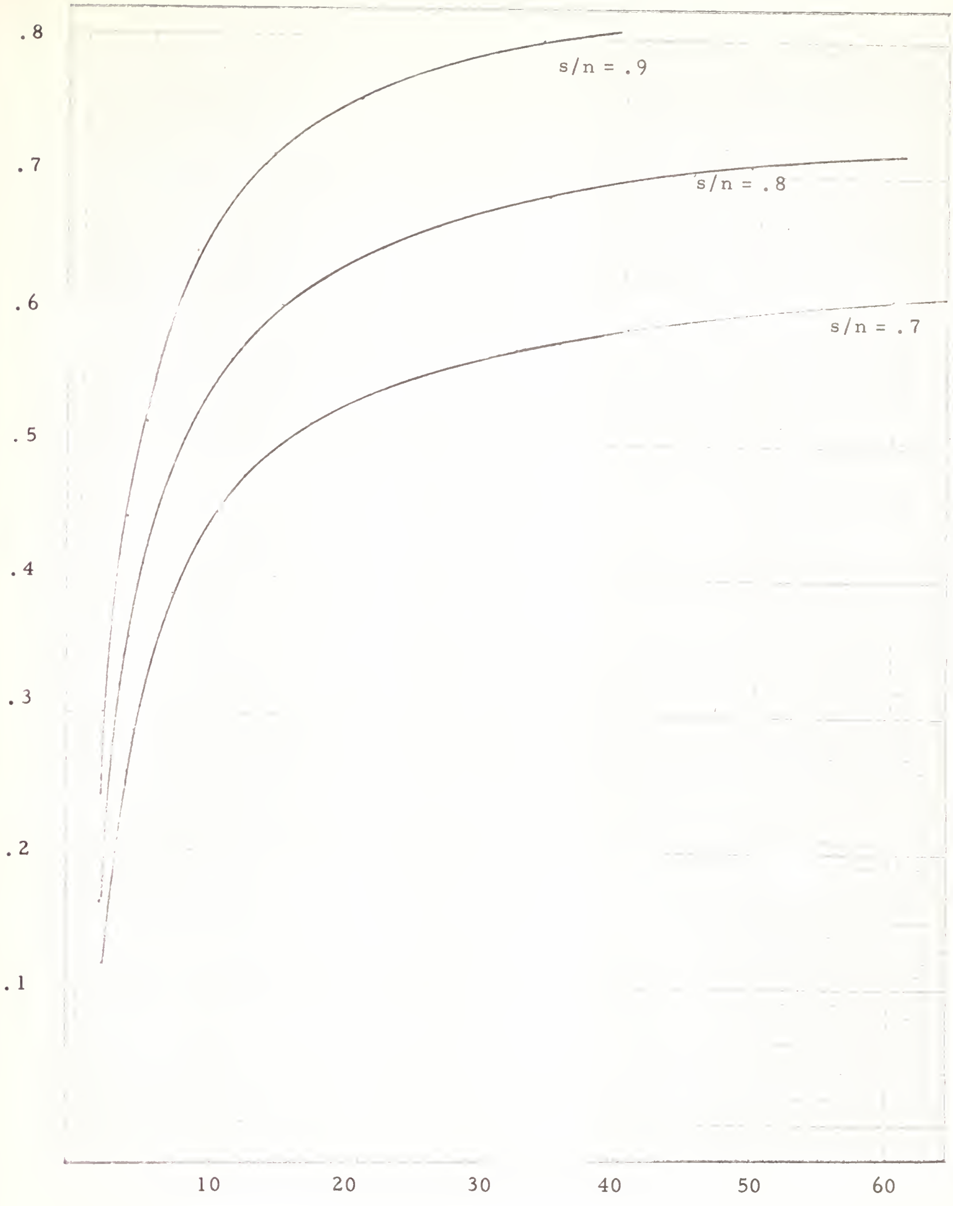
2.6 Comparison of Models One, Two, and Three

The sum of the variances for all three models was minimum for sample plan six. Sample plan six calls for extremely heavy testing in the early years and very little testing in the out years. Sample plan four results in values of the sum of the variances that are next smallest to the values for plan six. However, plan four is very similar to plan six. Note that plan three also calls for heavy testing in the early years and moderate testing in the out years. Sample plan three is different from plans four and six in that the testing in the early years, although heavy, is more moderate than that called for in the early years by plan six or four. The testing required in the out years by plans four and three is moderate. However, the scheduled testing is not as moderate as for sample plan six. For all cases, i.e., all models and values of the unknown parameters, the sample plans that call for moderate testing in the early years and heavy testing in the out years (plans two and five) result in values of the sum of the variances that are a maximum.

Testing heavily in the early years rather than testing uniformly throughout the years or testing heavily in the out years has some appeal to the practical. Since missile systems are programmed for an operational life of about ten years, planners might be reluctant to launch any large correction program determined necessary by testing

in the out years. Rather than spending money on a retiring system the planners might prefer to spend the money speeding up the phase in of the successor system. However, efforts along this line are usually not successful and in all likelihood an unsuspected discrepancy in a missile system that shows up in the out years will not be corrected.

Heavy testing conducted in the early years will allow for much higher confidence in the predictions of reliability for the out years. From these predictions, a potential discrepancy in the system that might not be noticed until the out years under other sample plans may be uncovered early in the system's life. Because of this, corrective action can be taken while time is available and planners are more willing to invest in the system.



90 PERCENT LOWER CONFIDENCE LIMIT
VERSUS SIZE OF TEST (n)

FIGURE 3.1.1

3. ANALYSIS AND DISCUSSION

3.1 Mathematical Support of the Inventory Model

In the final analysis, the number of missiles from a system counted on by the strategic planner will be determined by the number of missiles available and the 90 percent lower confidence limit for system reliability. In other words, if the 90 percent lower confidence limit of reliability is .7 and there are 200 missiles available, the strategic planner will count on .7(200) = 140 missiles.

When conducting an OT or FOT of a missile system, a missile is fired under the most realistic conditions attainable in peacetime and the impact of the missile on the test range is monitored to determine if the missile firing obtained the desired results. Whenever a missile is tested, the lower confidence limit of reliability comes a little closer to the observed success rate.[1] In fact, if all the available missiles were tested, then the lower confidence limit for reliability would be the observed success rate. A plot of the 90 percent lower confidence limit versus the number tested for observed success rates of .7, .8, and .9 is shown in Figure 3.1.1.

The crux of determining the number of missiles to test is depicted here. Obviously, we want the lower confidence limit of reliability to

[1] The observed success rate is merely the number of successful firings divided by the total number of firings.

be close to the real reliability of the system. However, to make the lower confidence limit for system reliability the same as true system reliability, we must test all the missiles. Certainly this is not the solution. The problem of where to stop testing could be solved by observing the graphs in Figure 3.1.1 and noting that one is beyond the "knee"[1] of the curve if testing is conducted until the lower confidence limit is within about .1 or .09 of the observed success rate.

Let us investigate just what effect a criterion such as this will have on a fixed inventory of missiles. In this discussion, the following common notation will be used:

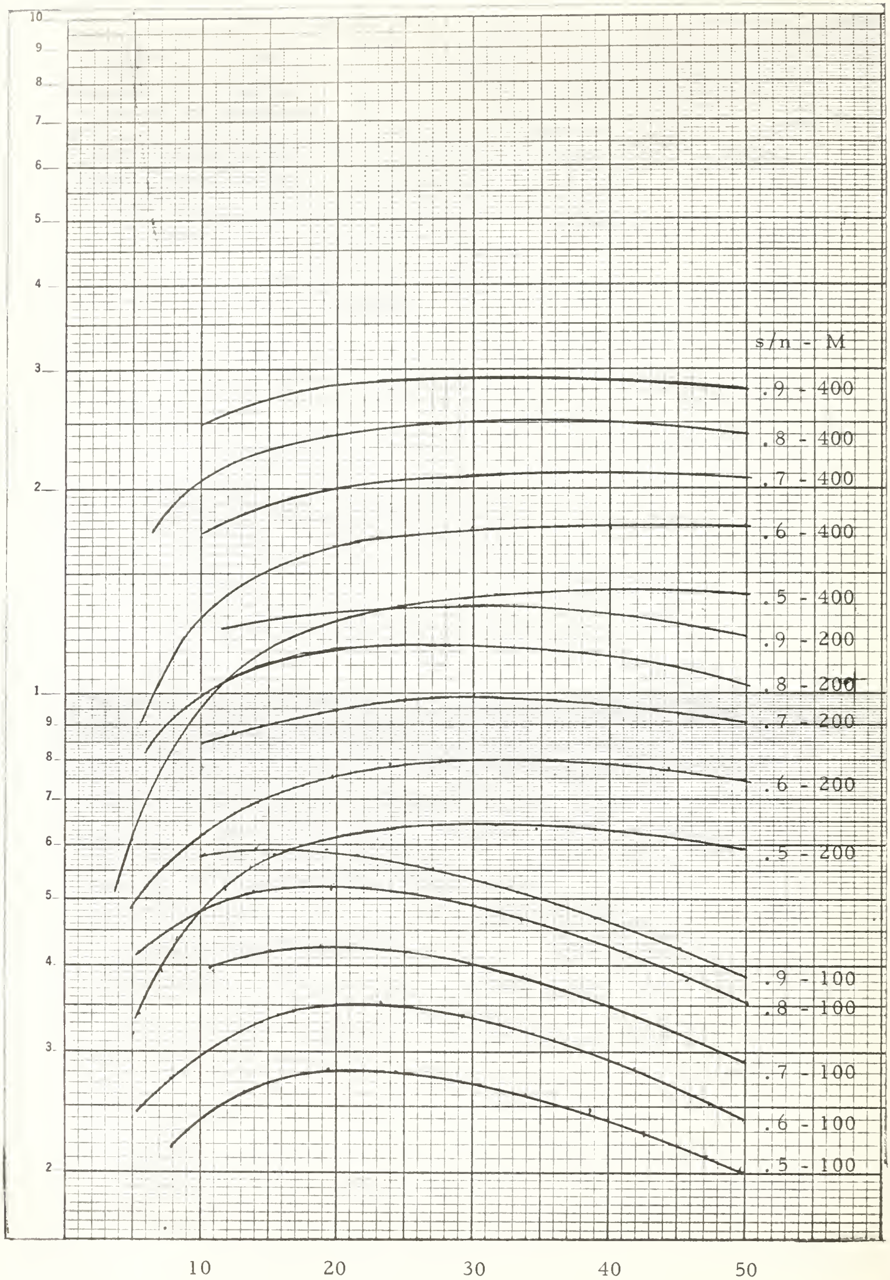
M = the total number of missiles in inventory;

n = the size of the test;

L = the 90 percent lower confidence limit for reliability. This number is determined by the size of the test and the observed success rate.

Assume that M is 200 and that the observed success rate of the missiles tested is running in the vicinity of .8. To satisfy the criterion that the lower confidence limit for reliability be within .09 of the observed success rate, i.e., .71, will require approximately 50 tests and the strategic planner will use (200-50) .71 = 106 as the expected number of

[1] The "knee" of the curve is a term used commonly by some economists and is loosely defined as the point on the curve where the marginal increase in utility per resource expenditure is relatively low.



VALUE OF L(M - n) VERSUS n

FIGURE 3.1.2

reliable missiles. However, if, under the same circumstances (i.e., observed success rate of .8) the system was tested only 30 times, the lower confidence limit for reliability would be .675 and the strategic planner could count on .675(200-30) = 114 as the expected number of reliable missiles. Not only is this a gain in the expected number of reliable missiles in inventory, but it is also a monetary gain of the cost of testing 20 missiles. Perhaps one more such excursion will illuminate more fully the costs of the criterion outlined above. Assume that M is 150 and the observed success rate is running about .7. To satisfy the criterion that testing be continued until the 90 percent lower confidence limit on reliability is within .1 of the observed success rate, i.e., a lower confidence limit of .6, will require 50 tests and the strategic planner will count on .6(150-50) = 60 as the expected number of reliable missiles. Under the same circumstances, if only 30 missiles had been tested, the lower confidence limit for reliability would be .565 and the strategic planner would count on .565(150-30) = 68 missiles. Again this is a savings of eight more missiles in the expected number of reliable missiles in inventory and the cost of testing 20 additional missiles.

For the reasons noted in the last paragraph, an alternate criterion for determining the number of missiles to test might be to choose n to maximize the quantity Q in the following expression.

$$Q = L(M - n)$$

Notice that when n is zero, L is zero, and therefore Q is zero. Whenever n is between zero and M, Q is positive. Since Q is zero when n equals M, we know that Q has a maximum value for some values of n in the interval 0 - M. In an effort to show that many complex problems do not require extensive and high-powered mathematics, the value of n that maximizes Q has been solved graphically.

By using semi-log paper and using values of L(M - n) as ordinates versus values of n along the abscissa, a plot of the values of Q for various success rates and values of M can be obtained. This graph is shown in Figure 3.1.2. Plots of the value of Q for all combinations of M = 50, 100, 200, and 400 and success rates of .5, .6, .7, .8, .9 are displayed. Notice that for values of M greater than 100 the value of n that universally maximizes Q is from 33 to 38 missiles. Thus, even though the success rates will be unknown prior to experimentation, it so happens that the optimum sample size n is basically a function only of M and the confidence level. (This can be seen from Figure 3.1.2.) Consequently, the following procedure can be implemented. This procedure is that for M between 200 and 400 and success rates greater than .5, the test size should be 33. For information, a table demonstrating the small increase in the 90 percent lower confidence limit for variations in test size about this "optimal" number is shown on the following page.

Observed Success Rate	Lower Confidence Limits for Test of		
	39 missiles	35 missiles	33 missiles
.7	.59	.59	.58
.8	.70	.69	.68
.9	.81	.80	.79

ONE-SIDED 90 PERCENT CONFIDENCE LIMITS

FIGURE 3.1.3

3.2 Description of Missiles Deployed Model

One problem with the model presented in the first section is that we can not expect to use all of the missiles in inventory. More than likely the Polaris system will use only the missiles that are already deployed at the outbreak of the exchange.

An appealing model that could be used to determine the optimal test size, considering only use from deployed missiles, can be constructed if it is assumed that our defense resources have been allocated in a near optimal manner. In other words, money was spent on the Polaris program until the cost of increasing our defense posture through Polaris was more than the cost of increasing our defense posture by an equal amount using another system. Realize now that testing a system merely allows us to count on a larger number of reliable missiles.[1] The expected number of reliable missiles can also be increased by procuring a larger number of systems. The cost of increasing the expected number of reliable missiles through procurement is called the marginal cost of reliable missiles through procurement. The marginal cost of increasing the expected number of reliable missiles through testing can be computed using the following formula.

[1] Of course, if, on subsequent testing, the observed success rate was lower, the expected number of reliable missiles used by the planner could conceivably be lower in this case. In this exposition, we are assuming that success rate is nearly constant.

$$MC = \Delta L(N) / C$$

where:

- MC = The marginal cost of increasing the expected number of reliable missiles through testing.
- ΔL = The increase in the 90 percent lower confidence limit for reliability caused by the last test of the system.
- N = The average number of missiles deployed at any time.
- C = The cost of testing the missile.

Now testing should be conducted until the marginal cost of increasing the expected number of reliable missiles through testing is equal to the marginal cost of increasing the expected number of reliable missiles through procurement.

An immediate argument against this model is that to increase the number of reliable missiles through procurement would require a new submarine. This is an extremely costly process and would therefore lead to excessive missile testing if the above criterion was established. However, the dilemma can be resolved by considering the marginal cost of increasing the yield over the target by any system that competes with Polaris. The lowest cost in this group should be used in place of the marginal cost for Polaris. Thus, one could say that testing will stop whenever the cost of increasing the expected yield of

[1] Yield is introduced here merely to have a common ratio between Polaris and the systems competing with Polaris.

the Polaris system from testing equals the cost of increasing the expected yield of any competing system through procurement. In this case, MC would be computed as follows:

$$MC = \Delta L(N)(Y) / C$$

where Y is the yield from a Polaris missile.

An explicit solution for n is not included here because the marginal cost of procurement for Polaris and the systems competing with Polaris is not known to the author. However, these figures have been computed; and once they are used, an explicit value for optimal n follows immediately.

3.3 Mathematical Support of Model One

Consider a collection of N missiles that are characterized by lifetimes $X_1, X_2, \ldots, X_n$ where X_i represents the time from initial inspection and release until the i^{th} item deteriorates to an unacceptable state. If it is assumed that the initial states of the missiles are the same, then the X_i's are non-negative independent, identically distributed random variables. Suppose that there are ten distinct times that observations are to be taken from the collection of missiles. Also, assume that K_i items are to be observed at time t_i. These observations can be summarized by the random variables $Y_1, Y_2, Y_3, \ldots, Y_{K_i}$ where:

$$Y_i = \begin{cases} 1 & \text{if the missile system fails} \\ 0 & \text{if the missile system is successful} \end{cases}$$

On the basis of this information, it is desired to estimate R(t), P (a missile's lifetime is greater than t). The estimation procedure which is proposed is of the following form. For each time t, the estimate of R(t) is given by

$$R(t) = \frac{\prod_{i=1}^{t} N - n_{i-1} \quad - \sum_{K=(n_{i-1}+1)}^{n_i} Y_K}{N(N - n_1)(N - n_2) \ldots (N - n_{t-1})}$$

where:

$$n_0 = 0$$

where:

$$n_i = \sum_{r=1}^{i} K_r, \quad i = 1, 2, \ldots, 10$$

$$K_r = \text{no. of trials in year } r$$

From this, it is obtained at once that:

$$E[R(t)] = \frac{\prod_{i=1}^{t} [N - n_{i-1} - K_i(1 - p_i)]}{N(N - n_1)(N - n_2)\ldots(N - n_{t-1})}$$

$$E[R(t)^2] = \frac{\prod_{i=1}^{t} [(N - n_{i-1})^2 - 2(N - n_{i-1}) K_i(1 - p_i)}{[N(N - n_1)(N - n_2)\ldots(N - n_{t-1})]^2}$$

$$\frac{+ K_i p_i (1 - p_i) + K_i^2 (1 - p_i)^2]}{[N(N - n_1)(N - n_2)\ldots(N - n_{t-1})]^2}$$

and the variance of the estimator is given by

$$\text{Var } R(t) = E[R(t)^2] - [E[R(t)]]^2$$

Note that Model One can be defined as:

$$R(t) = \left[R(t - 1)\right]\left[1 - \frac{\text{no. of failures in the } t^{th} \text{ test}}{N - \text{no. of missiles tested prior to time } t}\right]$$

By considering numerous different possible results of testing, it was observed that this estimator tends to be more conservative than the Maximum Likelihood Estimator of reliability in the early years. In the out years, this estimator tends to be pessimistic. In addition to the above properties, this estimate lends continuity to the estimates of reliability. The excursions listed on the next page illustrate these points.

Excursion 1:

$N = 100$, $K_i = 10$, for all i. There are two failures in each test.

Year	Model One Estimate	MLE*
1	$\tilde{R}(t_1) = \frac{100 - 2}{100} = .98$	$\hat{R}(t_1) = \frac{10 - 2}{10} = .8$
2	$\tilde{R}(t_2) = .98(1 - \frac{2}{90}) = .956$	$\hat{R}(t_2) = .8$
3	$\tilde{R}(t_3) = .956(1 - \frac{2}{80}) = .932$	$\hat{R}(t_3) = .8$
4	$\tilde{R}(t_4) = .932(1 - \frac{2}{70}) = .904$	$\hat{R}(t_4) = .8$
5	$\tilde{R}(t_5) = .904(1 - \frac{2}{60}) = .874$	$\hat{R}(t_5) = .8$
6	$\tilde{R}(t_6) = .874(1 - \frac{2}{50}) = .837$	$\hat{R}(t_6) = .8$
7	$\tilde{R}(t_7) = .837(1 - \frac{2}{40}) = .795$	$\hat{R}(t_7) = .8$
8	$\tilde{R}(t_8) = .795(1 - \frac{2}{30}) = .741$	$\hat{R}(t_8) = .8$
9	$\tilde{R}(t_9) = .741(1 - \frac{2}{20}) = .667$	$\hat{R}(t_9) = .8$
10	$\tilde{R}(t_{10}) = .667(1 - \frac{2}{10}) = .532$	$\hat{R}(t_{10}) = .8$

* Estimate of Reliability using only the yearly test information.

Excursion 2:

$N = 100, \quad K_i = 10, \quad$ for all i.

Year	1	2	3	4	5	6	7	8	9	10
ΣY_i	3	1	2	1	4	2	0	3	1	3

Note that the average number of failures per year is two.

Year	Model One Estimate	MLE*
1	$\tilde{R}(t_1) = \frac{100 - 3}{100} = .97$	$\hat{R}(t_1) = .7$
2	$\tilde{R}(t_2) = .99\ (1 - \frac{1}{90}) = .979$	$\hat{R}(t_2) = .9$
3	$\tilde{R}(t_3) = .979\ (1 - \frac{2}{80}) = .954$	$\hat{R}(t_3) = .8$
4	$\tilde{R}(t_4) = .954\ (1 - \frac{1}{70}) = .94$	$\hat{R}(t_4) = .9$
5	$\tilde{R}(t_5) = .94\ (1 - \frac{4}{60}) = .88$	$\hat{R}(t_5) = .6$
6	$\tilde{R}(t_6) = .88\ (1 - \frac{2}{50}) = .835$	$\hat{R}(t_6) = .8$
7	$\tilde{R}(t_7) = .835$	$\hat{R}(t_7) = 1.0$
8	$\tilde{R}(t_8) = .835\ (1 - \frac{3}{30}) = .752$	$\hat{R}(t_8) = .7$
9	$\tilde{R}(t_9) = .752\ (1 - \frac{1}{20}) = .713$	$\hat{R}(t_9) = .9$
10	$\tilde{R}(t_{10}) = .713\ (1 - \frac{3}{10}) = .5$	$\hat{R}(t_{10}) = .7$

* Estimate of Reliability using only the yearly test information.

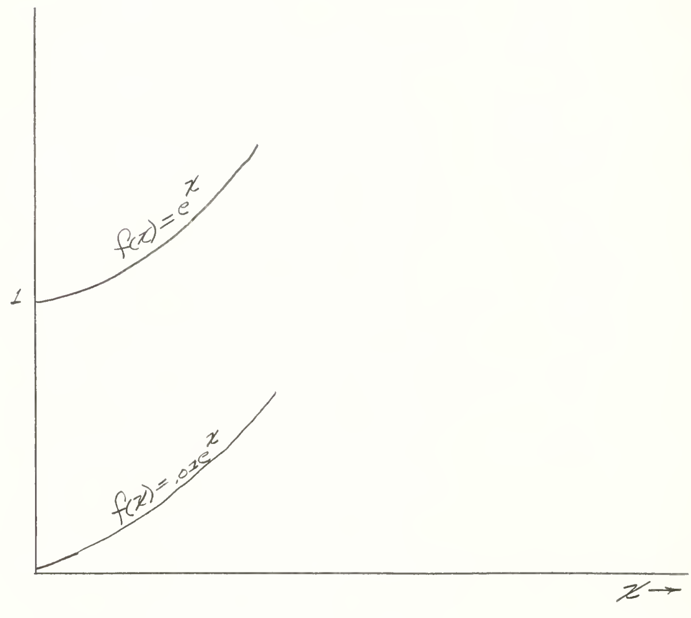
Excursion 3:

$N = 200$

Year	1	2	3	4	5	6	7	8	9	10
K	60	20	20	20	20	20	10	10	10	10
ΣY_i	9	3	4	3	2	5	0	3	4	4

Year	Model One Estimate	MLE*
1	$\tilde{R}(t_1) = \frac{200 - 9}{200} = .955$	$\hat{R}(t_1) = .85$
2	$\tilde{R}(t_2) = .955\,(1 - \frac{3}{140}) = .933$	$\hat{R}(t_2) = .85$
3	$\tilde{R}(t_3) = .933\,(1 - \frac{4}{120}) = .897$	$\hat{R}(t_3) = .8$
4	$\tilde{R}(t_4) = .897\,(1 - \frac{3}{100}) = .87$	$\hat{R}(t_4) = .85$
5	$\tilde{R}(t_5) = .87\,(1 - \frac{2}{80}) = .847$	$\hat{R}(t_5) = .9$
6	$\tilde{R}(t_6) = .847\,(1 - \frac{5}{60}) = .725$	$\hat{R}(t_6) = .75$
7	$\tilde{R}(t_7) = .725$	$\hat{R}(t_7) = 1.0$
8	$\tilde{R}(t_8) = .725\,(1 - \frac{3}{30}) = .652$	$\hat{R}(t_8) = .7$
9	$\tilde{R}(t_9) = .652\,(1 - \frac{4}{20}) = .522$	$\hat{R}(t_9) = .6$
10	$\tilde{R}(t_{10}) = .652\,(1 - \frac{4}{10}) = .39$	$\hat{R}(t_{10}) = .6$

* Estimate of Reliability using only the yearly test information.



GRAPH OF e^x VERSUS x

FIGURE 3.4.1

3.4 Mathematical Support of Model Two

The exponential decay model has been used on numerous occasions to describe reliability decay of electronic systems. To this end, Model Two introduces an exponential decay. However, to avoid the rapid decrease of reliability in the first years that ensues using the exponential decay form $R(t) = p_0 e^{-ct}$, the following expression for reliability is introduced:

$$R(t) = p_0 - Be^{ct}$$

where:

p_0 = the system's initial reliability;

B = a known constant;

c = an unknown parameter;

t = time, i.e., 1, 2, ..., 10.

Figure 3.4.1 illustrates that e^{ct} is greater than or equal to one for all positive values of c and t. Therefore, if a value of B that is greater than or equal to one is chosen, negative values of reliability will result. To avoid this, B has been arbitrarily set equal to .01. This value merely displaces downward the value of the second term in this expression. With this value of B, the expression for reliability will be positive for all combinations of t from zero to ten and c from .00424 to .424.

Because of the invariance principle, we may obtain an estimator

for c, call it $\hat{c}$; and then the estimate for reliability can be obtained by substituting $\hat{c}$ for c. We will, therefore, assume that p_0 is known and obtain a Maximum Likelihood Estimator for c. As a reminder, all testing done in year i is designated as test i. Each individual firing of a missile is called a trial. K_i is the number of trials in the i^{th} test, and s_i is the number of successful trials in test i.

The MLE is obtained in the following manner. The probability that s successes are observed during the i^{th} test can be expressed by:

$$P[\text{no. of successes} = s] = \binom{K_i}{s} (R(t))^s (1 - R(t))^{K_i - s}$$

$$= \binom{K_i}{s} (p_0 - Be^{ct})^s (1 - p_0 + Be^{ct})^{K_i - s}$$

Let $L(c) = P[\text{no. of successes} = s]$. Then

$$\ln L(c) = \ln \binom{K_i}{s} + s \ln (p_0 - Be^{ct}) + (K_i - s) \ln (1 - p_0 + Be^{ct})$$

$$\frac{\partial \ln L(c)}{\partial c} = \frac{-stBe^{ct}}{p_0 - Be^{ct}} + \frac{(K_i - s)(tBe^{ct})}{1 - p_0 + Be^{ct}}$$

To obtain a Maximum Likelihood Estimator of c, set this equation equal to zero. The solution to this equation is $\hat{c}$. Doing this yields

$$\hat{c} = \frac{1}{t} \ln \frac{K_i p_0 - s_i}{K_i B}$$

expanding in a Taylor series

$$\hat{c} = \frac{1}{t}\ln\left[\frac{p_0}{B} - \frac{s_i}{BK_i}\right] = \frac{1}{t}\ln\left[\frac{p_0}{B}\left(1 - \frac{\hat{p}_i}{p_0}\right)\right] = \frac{1}{t}\left\{\ln\frac{p_0}{B} - \left[-\ln\left(1 - \frac{\hat{p}_i}{p_0}\right)\right]\right\}$$

$$= \frac{1}{t}\ln\frac{p_0}{B} - \frac{1}{t}\left\{\frac{\hat{p}_i}{p_0} + \frac{\hat{p}_i^{\,2}}{2p_0^{\,2}} + \frac{\hat{p}_i^{\,3}}{3p_0^{\,3}} + \dots\right\}$$

$$\hat{c}^2 = \frac{1}{t^2}\ln\frac{p_0}{B} - \frac{2}{t^2}\ln\frac{p_0}{B}\left\{\frac{\hat{p}_i}{p_0} + \frac{\hat{p}_i^{\,2}}{2p_0^{\,2}} + \frac{\hat{p}_i^{\,3}}{3p_0^{\,3}} + \dots\right\}$$

$$+ \frac{1}{t^2}\left\{\frac{\hat{p}_i^{\,2}}{p_0^{\,2}} + \frac{\hat{p}_i^{\,3}}{p_0^{\,3}} + \frac{11\hat{p}_i^{\,4}}{12p_0^{\,4}} + \frac{1}{3}\frac{\hat{p}_i^{\,5}}{p_0^{\,5}} + \frac{1}{9}\frac{\hat{p}_i^{\,6}}{p_0^{\,6}} + \dots\right\}$$

Now since $\hat{p}^r = \frac{s_i^{\,r}}{K_i^{\,r}}$ then $E\left[\hat{p}_i^{\,r}\right] = \frac{1}{K_i^{\,r}}E\left[s_i^{\,r}\right]$. Now s is a binomial random variable with parameters p_i and K_i. The method of obtaining the expected values of the powers of s using factorial moments is illustrated in Appendix 1.

Using these values, the mean and variance of the estimator for c may be approximated from the following expressions.[1]

[1] For ease in writing these long equations, p_i has been used for R(t).

$$E[\hat{c}_i] = \frac{1}{t_i} \ln \frac{p_0}{B} - \frac{1}{t_i} \left\{ \frac{1}{K_i p_0} E[s_i] + \frac{1}{2K_i^2 p_0^2} E[s_i^2] + \frac{1}{3K_i^3 p_0^3} E[s_i^3] + \dots \right\}$$

$$= \frac{1}{t_i} \ln \frac{p_0}{B} - \frac{1}{t_i} \left\{ \frac{1}{K_i p_0} K_i p_i + \frac{1}{2K_i^2 p_0^2} [K_i(K_i - 1) p_i^2 + K_i p_i] + \frac{1}{3K_i^3 p_0^3} [K_i(K_i - 1)(K_i - 2) p_i^3 + 3(K_i(K_i - 1) p_i^2 + K_i p_i) - 2(K_i p_i)] + \dots \right\}$$

$$E[\hat{c}_i^2] \doteq \frac{1}{t_i^2} \left\{ \ln^2 \frac{p_0}{B} - 2 \ln \frac{p_0}{B} \left[\frac{1}{K_i p_0} E[s_i] + \frac{1}{2K_i^2 p_0^2} E[s_i^2] + \frac{1}{3K_i^3 p_0^3} E[s_i^3] \right] + \frac{1}{K_i^2 p_0^2} E[s_i^2] + \frac{1}{K_i^3 p_0^3} E[s_i^3] + \frac{11}{12K_i^4 p_0^4} E[s_i^4] + \frac{1}{3K_i^5 p_0^5} E[s_i^5] + \frac{1}{9K_i^6 p_0^6} E[s_i^6] + \dots \right\}$$

$$E[\hat{c}_i^{\,2}] = \frac{1}{t_i^{\,2}} \Big\{ \ln^2 \frac{p_0}{B} - 2 \ln \frac{p_0}{B} \, \{ \frac{1}{K_i p_0} \, (K_i p_i)$$

$$+ \frac{1}{2K_i^{\,2} \, p_0^{\,2}} \, [K_i \, (K_i - 1) \, p_i^{\,2} + K_i \, p_i]$$

$$+ \frac{1}{3K_i^{\,3} \, p_0^{\,3}} \, [K_i (K_i - 1) \, (K_i - 2) \, p_i^{\,3} + 3[K_i (K_i - 1) p_i^{\,2} + K_i p_i]$$

$$- 2 K_i \, p_i \,] + \ldots \}$$

$$+ \frac{1}{K_i^{\,2} \, p_0^{\,2}} \, (K_i \, (K_i - 1) \, p_i^{\,2} + K_i \, p_i)$$

$$+ \frac{1}{K_i^{\,3} \, p_0^{\,3}} \, [K_i (K_i - 1) \, (K_i - 2) \, p_i^{\,3}$$

$$+ 3 \, [K_i \, (K_i - 1) \, p^2 + K_i \, p_i] - 2 K_i p_i]$$

$$+ \frac{11}{12K_i^{\,4} \, p_0^{\,4}} \, \{K_i \, (K_i - 1) \, (K_i - 2) \, (K_i - 3) p_i^{\,4}$$

$$+ 6 \, [K_i \, (K_i - 1) \, (K_i - 2) \, p_i^{\,3} + 3[K_i \, (K_i - 1) \, p_i^{\,2} + K_i \, p_i]$$

$$- 2 K_i p_i \,] - 11 \, [K_i \, (K_i - 1) \, p_i^{\,2} \,]$$

$$+ 6 K_i p_i \, \} + \ldots \Big\}$$

$$\mathrm{Var} \, [\hat{c}_i^{\,2}] = E \, [\hat{c}_i^{\,2}] - E^2 \, [\hat{c}_i]$$

3.5 Mathematical Support of Model Three

Models One and Two involve fairly complicated expressions for $R(t)$. It might be illuminating to compare these more complicated models with a simple linear model. Therefore, the following expression for reliability is proposed.

$$R(t) = p_0 - at$$

where:

p_0 = the initial reliability of the system;

a = an unknown parameter;

t = time, i.e., 1, 2, ..., 10.

A Maximum Likelihood Estimator for a will, by the invariance principle, result in a Maximum Likelihood Estimator for reliability. The MLE of a is found in the following manner. The probability that s successes are observed in K_i trials can be expressed as:

$$P[s_i = s] = \binom{K_i}{s} (R(t))^s (1 - R(t))^{K_i - s}$$

$$= \binom{K_i}{s} (p_0 - at)^s (1 - p_0 + at)^{K_i - s}$$

Calling this expression $L(a)$ and taking logs:

$$\ln L(a) = \ln \binom{K_i}{s} + s \ln (p_0 - at) + (K_i - s) \ln (1 - p_0 + at)$$

and

$$\frac{\partial \ln L(a)}{\partial a} = \frac{st}{p_0 - at} + \frac{(K_i - s)\, t}{1 - p_0 + at}$$

Setting this expression equal to zero and solving for a gives

$$\hat{a} = \frac{K_i p_0 - s}{K_i t} = \frac{p_0}{t} - \frac{s}{K_i t_i}$$

and

$$E[\hat{a}] = a \qquad \mathrm{Var}[\hat{a}] = \frac{1}{K_i t^2}(p_0 - at)(1 - p_0 + at)$$

follows directly from the expression for $\hat{a}$.

APPENDIX I

FACTORIAL MOMENTS OF A BINOMIAL RANDOM VARIABLE

The moments of a binomial random variable, s, may be generated using the probability generating function

$$M_s(z) = (q + pz)^K$$

where K indicates the number of trials and

$$q = 1 - p \ .$$

For simplicity in notation, let

$$G = M_s(z)$$

and

$$E[s^K] = \overline{s}^K \ .$$

Now

$$\left.\frac{\partial G}{\partial z}\right|_{z=1} = K(q + pz)^{K-1} p \Big|_{z=1} = Kp$$

$$\left.\frac{\partial^2 G}{\partial z^2}\right|_{z=1} = K(K-1)(q + pz)^{K-2} p^2 \Big|_{z=1} = K(K-1)p^2 \ .$$

Since the term $(q + pz)^x$ will always reduce to one when z is set equal to one, we see a recursion relation, namely:

$$\left.\frac{\partial^n G}{\partial z^n}\right|_{z=1} = K(K-1) \ . \ . \ . \ (K - (n+1))\, p^n \ .$$

We may now deduce that:

$$E[s(s-1)\ldots(s-(n+1))] = K(K-1)\ldots(K-(n+1))\,p^{n}\,.$$

Proceeding from here, the expected values are determined below.

$$E[s(s-1)(s-2)] = K(K-1)(K-2)\,p^{3}$$

$$\bar{s}^{3} - 3\bar{s}^{2} + 2\bar{s} = K(K-1)(K-2)\,p^{3}$$

$$\therefore\ \bar{s}^{3} = K(K-1)(K-2)\,p^{3} + 3\bar{s}^{2} - 2\bar{s}$$

$$E[s(s-1)(s-2)(s-3)] = K(K-1)(K-2)(K-3)\,p^{4}$$

$$\bar{s}^{4} - 6\bar{s}^{3} + 11\bar{s}^{2} - 6\bar{s} = K(K-1)(K-2)(K-3)\,p^{4}$$

$$\bar{s}^{4} = K(K-1)(K-2)(K-3)\,p^{4} + 6\bar{s}^{3} - 11\bar{s}^{2} + 6\bar{s}$$

$$E[s(s-1)(s-2)(s-3)(s-4)] = K(K-1)(K-2)(K-3)(K-4)\,p^{5}$$

$$\bar{s}^{5} - 10\bar{s}^{4} + 35\bar{s}^{3} - 50\bar{s}^{2} + 24\bar{s} = K(K-1)(K-2)(K-3)(K-4)\,p^{5}$$

$$\bar{s}^{5} = K(K-1)(K-2)(K-4)\,p^{5} + 10\bar{s}^{4} - 35\bar{s}^{3} + 50\bar{s}^{2} - 24\bar{s}$$

$$E[s(s-1)(s-2)(s-3)(s-4)] = K(K-1)(K-2)(K-3)(K-4)(K-5)\,p^{6}$$

$$\bar{s}^{6} - 15\bar{s}^{5} + 85\bar{s}^{4} - 225\bar{s}^{3} + 274\bar{s}^{2} - 120\bar{s}$$

$$= K(K-1)(K-2)(K-3)(K-4)(K-5)\,p^{6}$$

$$\bar{s}^{6} = K(K-1)(K-2)(K-3)(K-4)(K-5)\,p^{6}$$

$$+ 15\bar{s}^{5} - 85\bar{s}^{4} + 225\bar{s}^{3} - 274\bar{s}^{2} + 120\bar{s}$$

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```
      DIMENSION P(10,3),EP(11),EPP(11),EPS(11)
      COMMON/VAR1/AK(10,6)
      DO 1   J=1,3
      GO TO (18,9,110),J
   18 AN=200   $ GO TO 101
    9 AN=300   $ GO TO 101
  110 AN=400   $ GO TO 101
  101 DO 1 JM=1,3
      GO TO (2,3,4),JM
    2 AL=.01   $ GO TO 5
    3 AL=.1    $ GO TO 5
    4 AL=.2    $ GO TO 5
    5 DO 6 JN=1,10
    6 P(JN,JM)=EXPF(-AL*JN)
      CALL SAMPLE (AN)
      DO 1 JO=1,6
      DO 8 JP=1,10
      M=JP-1
      IF(JP-1) 10,11,10
   11 EP(1)=((AN-AK(1,JO)*(1.-P(1,JM)))/AN)
      EPS(1) =EP(1)**2
      SUM = 0
      EPP(1) =(AN**2-2.*AN*AK(1,JO)*(1.-P(1,JM))+AK(1,JO)*P(1,JM)*(1.-P(
     11,JM))+(AK(1,JO)*(1.-P(1,JM)))**2)/AN**2
      SVAR =EPP(1)-EPS(1)
      PRINT 1002, EP(1), EPP(1),SVAR,AK(1,JO),P(1,JM)
 1002 FORMAT( 10X, E16.9,10X, E16.9, 10X, E16.9,10X,F4.0,10X,E9.8///)
      GO TO 8
   10 SUM=SUM+AK(M,JO)
      EP(JP) = EP(M)*(AN-SUM-AK(JP,JO)*(1.-P(JP,JM)))/(AN-SUM)
      EPS(JP) =EP(JP)**2
      EPP(JP) = EPP(M)*((AN-SUM)**2-2.*(AN-SUM)*AK(JP,JO)*(1.-P(JP,JM))
     1+AK(JP,JO)*P(JP,JM)*(1.-P(JP,JM))+(AK(JP,JO)*(1.-P(JP,JM)))**2)/
     2(AN-SUM)**2
      SVAR=SVAR + EPP(JP)- EPS(JP)
      PRINT 1004, EP(JP), EPP(JP), SVAR, AK(JP,JO),SUM,P(JP,JM)
 1004 FORMAT (10X,E16.9,10X,E16.9,10X,E16.9,10X,F4.0,5X,F4.0,10X,E9.8///
     1)
    8 CONTINUE
      PRINT 1000,AL,AN,(AK(I,JO),I=1,10),SVAR
 1000 FORMAT ( 10X,9HLAMBDA = ,F4.3,10X,4HN = ,F4.0,///7H  K1 = ,F4.0,7H
     1  K2 = ,F4.0,7H  K3 = ,F4.0,7H  K4 = ,F4.0,7H  K5 = ,F4.0,7H  K6 =
     2 ,F4.0,7H  K7 = ,F4.0,7H  K8 = ,F4.0,7H  K9 = ,F4.0,7H  K10= ,F4.0
     3,///10X, 19HSUM OF VARIANCES = ,E16.9////)
    1 CONTINUE
      END
```

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```
      SUBROUTINE SAMPLE (AN)
      COMMON/VAR1/AK(10,6)
      DO 100 JA=1,6
      GO TO (110,120,130,140,150,160),JA
  110 DO 111 K=1,10
  111 AK(K,JA)=.1*AN
      GO TO 100
  120 DO 121 KA=1,5
  121 AK(KA,JA)=.05*AN
      DO 122 KA=6,10
  122 AK(KA,JA)=.15*AN
      GO TO 100
  130 DO 131 KB=1,5
  131 AK(KB,JA)=.15*AN
      DO 132 KB=6,10
  132 AK(KB,JA)=.05*AN
      GO TO 100
  140 AK(1,4)=.3*AN
      DO 141 KC=2,6
  141 AK(KC,JA)=.1*AN
      DO 142 KC=7,10
  142 AK(KC,JA)=.05*AN
      GO TO 100
  150 DO 151 KD=1,4
  151 AK(KD,JA)=.05*AN
      AK(5,5)=.3*AN
      DO 152 KD=6,10
  152 AK(KD,JA)=.1*AN
      GO TO 100
  160 AK(1,6)=.3*AN    $ AK(2,6)=.2*AN
      DO 161 KE=3,6
  161 AK(KE,JA)=.1*AN
      AK(7,6)=.03*AN   $ AK(8,6)=.03*AN    $ AK(9,6)=.02*AN   $AK(10,6)=
     1.02*AN
  100 CONTINUE
      RETURN
      END
```

```
      DIMENSION IT(12),LABEL(6)
      COMMON/VAREI/AK(10,6)
      NEWPLOT=10
      LABEL(1)=4H   1
      LABEL(2)=4H   2
      LABEL(3)=4H   3
      LABEL(4)=4H   4
      LABEL(5)=4H   5
      LABEL(6)=4H   6
      DO 2001 NON=1,3
      GO TO (2003,2004,2005),NON
 2003 CP=.7
      GO TO 2006
 2004 CP=.8
      GO TO 2006
 2005 CP=.9
      GO TO 2006
 2006 DO 13 MI=1,3
      GO TO (7,8,9),MI
    7 AN=100
      CALL SAMPLE(AN)
      GO TO 4001
    8 AN=200
      CALL SAMPLE(AN)
      GO TO 4001
    9 AN=400
      CALL SAMPLE(AN)
      GO TO 4001
 4001 DO 112 I=1,12
  112 IT(I)=8H
      IT(7)=8HP(0) =
      IT(8)=ICODE (CP)
      IT(10)=8HN =
      IT(11)=ICODE (AN)
      IT(1)=8HSUM OF V
      IT(2)=8HARIANCES
      IT(3)=8H VS A
      IT(4)=8H (DAVE S
      IT(5)=8HTANFORD)
      DO 133 J=1,6
      CALL PLOT(CP,     J,NON,NEWPLOT,IT,LABEL)
  133 CONTINUE
      NEWPLOT=10
   13 CONTINUE
 2001 CONTINUE
      END
```

```
      SUBROUTINE PLOT(CP,    J,NCN,NEWPLOT,IT,LABEL)
      COMMON/VARE1/AK(10,6)
      DIMENSION IT(12),LABEL(6),PAC(100),SVAR(100),VAR(10)
      DIMENSION PAC2(100,2),SVAR2(100,2)
      DIMENSION P(10)
      GO TO (1000,1002,1004),NCN
 1000 DO 100 N=1,100
      M=N-1
      IF(N-1) 110,120,110
  120 PAC(1) = .00424
      GO TO 4000
  110 PAC(N)=PAC(M)+.00424
      GO TO 4000
 4000 DO 20 IP=1,10
      FIP=IP
   20 P(IP) = CP-.01*EXPF(FIP*PAC(N))
      GO TO 102
  100 CONTINUE
      GO TO 1100
 1002 DO 1101 N=1,100
      M=N-1
      IF(N-1) 1102,1104,1102
 1104 PAC(1) =.00438
      GO TO 4000
 1102 PAC(N) =PAC(M) +.00438
      GO TO 4000
 1101 CONTINUE
      GO TO 1100
 1004 DO 2101 N = 1,100
      M=N-1
      IF(N-1)2102,2104,2102
 2104 PAC(1)=.00449
      GO TO 4000
 2102 PAC(N)=PAC(M) +.00449
      GO TO 4000
 2101 CONTINUE
      GO TO 1100
  102 DO 141 KK=1,10
      FKK=KK
      ES=AK(KK,J)*P(KK)
      ES2=AK(KK,J)*(AK(KK,J)-1.)*P(KK)**2 +ES
      ES3= AK(KK,J)*(AK(KK,J)-1.)*(AK(KK,J)-2.)*P(KK)**3 +3.*ES2-2.*ES
      ES4=AK(KK,J)*(AK(KK,J)-1.)*(AK(KK,J)-2.)*(AK(KK,J)-3.)*P(KK)**4 .
     1 +6.*ES3 -11.*ES2 + 6.*ES
      ES5 = AK(KK,J)*(AK(KK,J)-1.)*(AK (KK,J)-2.)*(AK(KK,J)-3.)*(
     1 AK(KK,J)-4.)*P(KK)**5 + 10.*ES4- 35.*ES3 +50.*ES2-24.*ES
      ES6 = AK(KK,J)*(AK(KK,J)-1.)*(AK (KK,J)-2.)*(AK(KK,J)-3.)*(AK(KK
     1 )-4.)*(AK(KK,J)-5.)*P(KK)**6 +15.*ES5-85.*ES4+225.*ES3
     2 -274.*ES2 +120.*ES
      DENOM=AK(KK,J)*CP
      CPL =LOGF(CP/.01)
      EC=(1./FKK*CPL -1./FKK*(1./DENOM*ES +1./(2.*DENOM**2)*ES2
     1 +1./(3.*DENOM**3)*ES3))**2
      EC2=1./FKK**2*(CPL**2-2.*CPL*(1./DENOM*ES +1./ (2.*DENOM**2)*ES2
     1 +1./(3.*DENOM**3)*ES3) +1./(DENOM**2)*ES2+1./(DENOM**3)
     2 *ES3 + 11./(12.*DENOM**4)*ES4 +1./(5.*DENOM**5)*ES5
```

```
     3 + 1./(9.*DENOM**6)*ES6)
       VAR=EC2-EC
       IF(KK-1) 131,133,131
   133 SVAR(N)=VAR
       GO TO 141
   131 SVAR(N) = SVAR(N) +VAR
   141 CONTINUE
       GO TO (100,1100,2100 )NOM
  1100 IF(NEWPLOT-10)1,2,1
     2 NEWPLOT=0
       II=0
       MODCURV=1
       NUMPTS=100
       KOUNT=1
       GO TO 3
     1 MODCURV=2
       NUMPTS=NUMPTS-1
       KOUNT=KOUNT+1
       IF(KOUNT-2)101,109,101
   101 IF(KOUNT-5)106,109,106
   109 II=II+1
       DO 104I=1,100
       PAC2(I,II)=PAC(I)
   104 SVAR2(I,II)=SVAR(I)
       GO TO 200
   106 IF (KOUNT-6)3,5,3
     5 MODCURV=3
     3 CALL DRAW(NUMPTS,PAC,SVAR,MODCURV,0,LABEL(KOUNT),IT,0,0,
      10,0,0,0,7,9,1,LAST)
       IF(KOUNT-6)200,201,200
   201 NUMPTS =100
       CALL DRAW(NUMPTS,PAC2(1,1),SVAR2(1,1),1,0,LABEL(2),IT,0,0,
      10,0,0,0,7,9,1,LAST)
       CALL DRAW(NUMPTS,PAC2(1,2),SVAR2(1,2),3,0,LABEL(5),IT,0,0,
      10,0,0,0,7,9,1,LAST)
   200 END
```

```
      SUBROUTINE SAMPLE (AN)
      COMMON/VAR21/AK([illegible],6)
      DO 100 JA=1,6
      GO TO (110,120,130,140,150,160),JA
  110 DO 111 K=1,10
  111 AK(K,JA)=.1*AN
      GO TO 100
  120 DO 121 KA=1,5
  121 AK(KA,JA)=.[illegible]*AN
      DO 122 KA=6,10
  122 AK(KA,JA)=.15*AN
      GO TO 100
  130 DO 131 KB=1,5
  131 AK(KB,JA)=.15*AN
      DO 132 KB=6,10
  132 AK(KB,JA)=.05*AN
      GO TO 100
  140 AK(1,4)=.1*AN
      DO 141 KC=2,6
  141 AK(KC,JA)=.1*AN
      DO 142 KC=7,10
  142 AK(KC,JA)=.05*AN
      GO TO 100
  150 DO 151 KD=1,4
  151 AK(KD,JA)=.[illegible]5*AN
      AK(5,5)=.3*AN
      DO 152 KD=6,10
  152 AK(KD,JA)=.1*AN
      GO TO 100
  160 AK(1,6)=.2*AN    $ AK(2,6)=.2*AN
      DO 161 KE=3,6
  161 AK(KE,JA)=.1*AN
      AK(7,6)=.05*AN   $ AK(8,6)=.05*AN   $ AK(9,6)=.02*AN   $AK(10,6)=
     1.02*AN
  100 CONTINUE
      RETURN
      END
```

```
      DIMENSION IT(12),LABEL(6)
      COMMON/VARE1/AK(10,6)
      NEWPLOT=10
      LABEL(1)=4H   1
      LABEL(2)=4H   2
      LABEL(3)=4H   3
      LABEL(4)=4H   4
      LABEL(5)=4H   5
      LABEL(6)=4H   6
      DO 2001 NON=1,3
      GO TO (2003,2004,2005),NON
 2003 CP=.7
      GO TO 2006
 2004 CP=.8
      GO TO 2006
 2005 CP=.9
      GO TO 2006
 2006 DO 13 MI=1,3
      GO TO (7,8,9),MI
    7 AN=200
      CALL SAMPLE(AN)
      GO TO 4001
    8 AN=300
      CALL SAMPLE(AN)
      GO TO 4001
    9 AN=400
      CALL SAMPLE(AN)
      GO TO 4001
 4001 DO 112 I=1,12
  112 IT(I)=8H
      IT(7)=8HP(0) =
      IT(8)=ICODE (CP)
      IT(10)=8HN =
      IT(11)=ICODE (AN)
      IT(1)=8HSUM OF V
      IT(2)=8HARIANCES
      IT(3)=8H VS A
      IT(4)=8H (DAVE S
      IT(5)=6HTANFORD)
      DO 133 J=1,6
      CALL PLOT(CP,    J,NON,NEWPLOT,IT,LABEL)
  133 CONTINUE
      NEWPLOT=10
   13 CONTINUE
 2001 CONTINUE
      END
```

```
      SUBROUTINE PLOT(CP,    J,NON,NEWPLOT,IT,LABEL)
      COMMON/VARE1/AK(10,6)
      DIMENSION IT(12),LABEL(6),PAC(100),SVAR(100),VAR(10)
      DIMENSION PAC2(100,2),SVAR2(100,2)
      GO TO (1000,1002,1004),NON
 1000 DO 100 N=1,100
      M=N-1
      IF(N-1) 110,120,110
  120 PAC(1)=.0007  $ GO TO 102
  110 PAC(N)=PAC(M)+.0007
  102 DO 141 NO=1,10
      FNO=NO
      VAR(NO)=1./(AK(NO,J) *FNO**2)*(CP-PAC(N)*FNO)*(1.-CP+PAC(N)*FNO
      IF(NO-1)130,140,130
  140 SVAR(N)=VAR(1)    $  GO TO 141
  130 SVAR(N)=SVAR(N)+VAR(NO)
  141 CONTINUE
  100 CONTINUE
      GO TO 1100
 1002 DO 1101 N=1,100
      M=N-1
      IF(N-1) 1102,1104,1102
 1104 PAC(1)=.0008   $ GO TO 1106
 1102 PAC(N)=PAC(M)+.0008
 1106 DO 1110  NO = 1,10
      FNO=NO
      VAR(NO)=1./(AK(NO,J)  *FNO**2)*(CP-PAC(N)*FNO)*(1.-CP+PAC(N)*FNO
      IF(NO-1) 1112,1114,1112
 1114 SVAR(N)=VAR(1)   $ GO TO 1110
 1112 SVAR(N)=SVAR(N)+VAR(NO)
 1110 CONTINUE
 1101 CONTINUE
      GO TO 1100
 1004 DO 2101 N = 1,100
      M=N-1
      IF(N-1)2102,2104,2102
 2104 PAC(1)=.0009    $ GO TO 2106
 2102 PAC(N)=PAC(M)+.0009
 2106 DO 2110 NO=1,10
      FNO=NO
      VAR(NO)=1./(AK(NO,J)  *FNO**2)*(CP-PAC(N)*FNO)*(1.-CP+PAC(N)*FNO
      IF(NO-1)2112,2114,2112
 2114 SVAR(N)=VAR(1)  $ GO TO 2110
 2112 SVAR(N)=SVAR  + VAR(NO)
 2110 CONTINUE
 2101 CONTINUE
      GO TO 1100
 1100 IF(NEWPLOT-10)1,2,1
    2 NEWPLOT=0
      II=0
      MODCURV=1
      NUMPTS=100
      KCOUNT=1
      GO TO 3
    1 MODCURV=2
      NUMPTS=NUMPTS-1
```

```
      KOUNT=KOUNT+1
      IF(KOUNT-2)101,109,101
  101 IF(KOUNT-5)106,109,106
  109 II=II+1
      DO 104I=1,100
      PAC2(I,II)=PAC(I)
  104 SVAR2(I,II)=SVAR(I)
      GO TO 200
  106 IF (KOUNT-6)3,5,3
    5 MODCURV=3
    3 CALL DRAW(NUMPTS,PAC,SVAR,MODCURV,0,LABEL(KOUNT),IT,0,0,
     10,0,0,0,7,9,1,LAST)
      IF(KOUNT-6)200,201,200
  201 NUMPTS =100
      CALL DRAW(NUMPTS,PAC2(1,1),SVAR2(1,1),1,0,LABEL(2),IT,0,0,
     10,0,0,0,7,9,1,LAST)
      CALL DRAW(NUMPTS,PAC2(1,2),SVAR2(1,2),3,0,LABEL(5),IT,0,0,
     10,0,0,0,7,9,1,LAST)
  200 END
```

```
      SUBROUTINE SAMPLE (AN)
      COMMON/VARE1/AK(10,6)
      DO 100 JA=1,6
      GO TO (110,120,130,140,150,160),JA
  110 DO 111 K=1,10
  111 AK(K,JA)=.1*AN
      GO TO 100
  120 DO 121 KA=1,5
  121 AK(KA,JA)=.05*AN
      DO 122 KA=6,10
  122 AK(KA,JA)=.15*AN
      GO TO 100
  130 DO 131 KB=1,5
  131 AK(KB,JA)=.15*AN
      DO 132 KB=6,10
  132 AK(KB,JA)=.05*AN
      GO TO 100
  140 AK(1,4)=.3*AN
      DO 141 KC=2,6
  141 AK(KC,JA)=.1*AN
      DO 142 KC=7,10
  142 AK(KC,JA)=.05*AN
      GO TO 100
  150 DO 151 KD=1,4
  151 AK(KD,JA)=.05*AN
      AK(5,5)=.3*AN
      DO 152 KD=6,10
  152 AK(KD,JA)=.1*AN
      GO TO 100
  160 AK(1,6)=.2*AN    $ AK(2,6)=.2*AN
      DO 161 KE=3,6
  161 AK(KE,JA)=.1*AN
      AK(7,6)=.03*AN  $ AK(8,6)=.03*AN   $ AK(9,6)=.02*AN  $AK(10,6)=
     1.02*AN
  100 CONTINUE
      RETURN
      END
```

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1. ORIGINATING ACTIVITY *(Corporate author)* U. S. Naval Postgraduate School Monterey, California	2a. REPORT SECURITY CLASSIFICATION UNCLASSIFIED 2b GROUP

3. REPORT TITLE

AN INDEPENDENT APPRAISAL OF THE POLARIS OPERATIONAL TESTING PROCEDURES

4. DESCRIPTIVE NOTES *(Type of report and inclusive dates)*

Master's Thesis

5. AUTHOR(S) *(Last name, first name, initial)*

Stanford, David L., Lieutenant, United States Navy

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
May, 1966	119	None

8a. CONTRACT OR GRANT NO. b. PROJECT NO. c. d.	9a. ORIGINATOR'S REPORT NUMBER(S) 9b. OTHER REPORT NO(S) *(Any other numbers that may be assigned this report)*

10. AVAILABILITY/LIMITATION NOTICES

(2) [redacted]

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY U. S. Naval Postgraduate School Monterey, California

13. ABSTRACT

The first section of this paper is concerned with determining the required lower confidence limit that must be met by testing after a missile system becomes operational. Some of the costs of making decisions about the required system reliability lower confidence limit are discussed. Two possible cost effective models for determining the optimum test size are suggested.

The second section of this paper is concerned with the effects of changing the number of missiles tested in each year while maintaining the total number of systems tests constant over the missile system's projected operating life. In other words, in this section, the effects of testing heavily in the first years versus testing heavily at the end of the system's life or versus testing uniformly throughout the life of the system are compared. For this comparison, the sum of the variances of the estimators determined from the results of the tests conducted in each year is obtained. This value is compared for six representative distributions of testing throughout an estimated system life of ten years.

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